ON THE LOWER ORDER TERMS OF THE ASYMPTOTIC EXPANSION OF ZELDITCH

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Contents

1. Introduction

A projective algebraic manifold M is a complex manifold in certain projective space CP^m, $m \geq \dim_{\mathbb{C}} M = n$. The hyperplane line bundle of CP^m restricts to an ample line bundle L on M, which is called a polarization on M. A Kähler metric g is called a polarized metric, if the corresponding Kähler form represents the first Chern class $c_1(L)$ of L in $H^2(M,\mathbb{Z})$. Given any polarized Kähler metric g, there is a Hermitian metric h on L whose Ricci form is equal to ω_g. For each positive integer $m > 0$, the Hermitian metric h induces a Hermitian metric h_m on L^m. Let $\{S_0^m, \cdots, S_{d_m-1}^m\}$ be an orthonormal basis of the space $H^0(M, L^m)$ of all holomorphic global sections of L^m. Such a basis $\{S_0^m, \cdots, S_{d_m-1}^m\}$ induces a holomorphic embedding φ_m of M into CP^{d_m-1} by assigning the point x of M to $[S_0^m(x), \cdots, S_{d_m-1}^m(x)]$ in CP^{d_m-1}. Let g_{FS} be the standard Fubini-Study metric on CP^{d_m-1}, i.e., $\omega_{FS} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\sum_{i=0}^{d_m-1}|w_i|^2$ for a homogeneous coordinate system $[w_0, \cdots, w_{d_m-1}]$ of CP^{d_m-1}. The $\frac{1}{m}$-multiple of g_{FS} on CP^{d_m-1} restricts to a Kähler metric $\frac{1}{m}\varphi_m^* g_{FS}$ on M. This metric is a polarized Kähler metric on M and is called the Bergman metric with respect to L.

One of the main theorem in [11] is the following

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Theorem (Tian). *With all the notations as above, we have*

$$||\frac{1}{m}\varphi_m^* g_{FS} - g||_{C^2} = O(\frac{1}{\sqrt{m}})$$

for any polarized metric on M with respect to L.

Using Tian's peak section method, Ruan [10] proved the C^∞ convergence and improved the bound to $O(\frac{1}{m})$. Recently, S. Zelditch [12] beautifully generalized the above theorem by using the Szegö kernel on the unit circle bundle of L^* over M. His result gives the asymptotic expansion of the potential of the Bergman metric:

Theorem (Zelditch). *Let M be a compact complex manifold of dimension n (over $\mathbb{C}$) and let $(L, h) \to M$ be a positive Hermitian holomorphic line bundle. Let x be a point of M. Let g be the Kähler metric on M corresponding to the Kähler form $\omega_g = Ric(h)$. For each $m \in \mathbb{N}$, h induces a Hermitian metric h_m on L^m. Let $\{S_0^m, \cdots, S_{d_m-1}^m\}$ be any orthonormal basis of $H^0(M, L^m)$, $d_m = \dim H^0(M, L^m)$, with respect to the inner product*

$$(S_1, S_2)_{h_m} = \int_M h_m(S_1(x), S_2(x))dV_g,$$

where $dV_g = \frac{1}{n!}\omega_g^n$ is the volume form of g. Then there is a complete asymptotic expansion:

$$\sum_{i=0}^{d_m-1} ||S_i^m(x)||_{h_m}^2 = a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \cdots \tag{1.1}$$

for certain smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely, for any k

$$||\sum_{i=0}^{d_m-1} ||S_i^m(x)||_{h_m}^2 - \sum_{j<R} a_j(x)m^{n-j}||_{C^k} \le C_{R,k}m^{n-R},$$

where $C_{R,k}$ depends on R, k and the manifold M.

Remark 1.1. The referee informed the author that Professor D. Catlin has also obtained the above Zelditch's result independently.

In this paper, we give a method to compute the coefficients $a_1(x), a_2(x), \cdots$ ($a_0(x) = 1$ was pointed out in [12] in a more general setting). Our result is

Theorem 1.1. *With the notations as in the above theorem, each coefficient $a_j(x)$ is a polynomial of the curvature and its covariant derivatives at x with weight j* [1]*. Such a polynomial can be found by finite many steps of algebraic*

[1] See Definition 2.1.

operations. In particular,

$$\begin{cases} a_0 = 1 \\ a_1 = \frac{1}{2}\rho \\ a_2 = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|Ric|^2 + 3\rho^2) \\ a_3 = \frac{1}{8}\Delta\Delta\rho + \frac{1}{24}\, div\, div\,(R, Ric) - \frac{1}{6} div\, div(\rho Ric) \\ +\frac{1}{48}\Delta(|R|^2 - 4|Ric|^2 + 8\rho^2) + \frac{1}{48}\rho(\rho^2 - 4|Ric|^2 + |R|^2) \\ +\frac{1}{24}(\sigma_3(Ric) - Ric(R,R) - R(Ric, Ric)), \end{cases}$$

where R, Ric and ρ represent the curvature tensor, the Ricci curvature and the scalar curvature of g, respectively and Δ represents the Laplace operator of M. For the precise definition of the terms in the expression of a_3, see Section 5.

We use the peak section method initiated by Tian in [11] to compute the coefficients $a_j (j \in \mathbb{N})$. Consider $H^0(M, L^m)$ for m large enough. Fixing a point $x \in M$, by the standard $\overline{\partial}$-estimate Tian observed that the sections which do not vanish at x at a very high order are known in the sense that one can completely control their behavior around the point x. These sections are called peak sections (in the terminology of [11]). We proved that the coefficients $a_1, a_2, \cdots$ only depend on the inner products of the peak sections. Various techniques are used to give the asymptotic expansion of these inner products, including some combinatorial lemmas, to simplify the computation and thus make the computation feasible.

Zelditch's work is based on the analysis of the asymptotic expansion of the Szegö kernel on the unit circle bundle of the ample line bundle over a complex manifold. To be more precise, let C be the unit circle bundle and let $\Pi(x, y)$ be its Szegö kernel (with the natural measure). Since C is S^1 invariant, we have Fourier coefficients

$$\Pi_m(x, x) = \int_{S^1} e^{-im\theta}\Pi(r_\theta x, x)d\theta,$$

where r_θ is the circle action. The key observation by Zelditch is that

$$\sum_{i=0}^{d_m - 1} ||S_i^m(x)||_{h_m}^2 = \Pi_m(x, x).$$

Thus the general theory of Szegö kernel can be applied.

There are a lot of works on the Bergman and Szegö kernels on the pseudoconvex domain on $\mathbb{C}^n$ ([3] [4], [1], [6], [7] and [8], for example) following the program of Fefferman [5]. While our method is completely complex-geometric, it should also be possible to compute the coefficients from the general theory of Szegö kernel. In particular, we noticed the works in [1] and [7], the coefficients are proved to be the Weyl functionals of the curvature tensor of the ambient metric defined by Fefferman. But I don't know how to relate it to the curvature of the base manifold.

The organization of the paper is as follows: in §2 we introduce the concept of peak sections initiated by Tian and discuss their properties; in §3 we give

the iteration process; in §4 we prove the main theorem of this paper except for the computation of the a_3 term; and in §5 and in the Appendix we calculate the a_3 term.

Acknowledgment. The idea that peak section method can be used to get the result is from G. Tian. The author thanks him for the encouragement and help during the preparation of this paper. He also thanks S. Zelditch for sending him the preprint [12] and bringing him to the attention of the works of K. Hirachi. The author thanks K. Hirachi, L. Boutet de Monvel and D. Phong for their help. Finally, the author deeply thanks the referee for the careful proofreading and many suggestions to improve the organization and the style of this paper.

2. Peak Global Sections

Let M be an n-dimensional algebraic manifold with a positive Hermitian line bundle $(L, h) \to M$. Suppose that the Kähler form ω_g is defined by the curvature $Ric(h)$ of h. That is, fixing a point $x_0 \in M$, under local coordinate $(z_1, \cdots, z_n)$ at x_0,

$$\omega_g = -\frac{\sqrt{-1}}{2\pi} \sum_{\alpha,\beta=1}^{n} \frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta} \log a \, dz_\alpha \wedge d\overline{z}_\beta = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha,\beta=1}^{n} g_{\alpha\overline{\beta}} dz_\alpha \wedge d\overline{z}_\beta,$$

where a is the local representation of the Hermitian metric h.

Let $d = d_m = \dim_{\mathbb{C}} H^0(M, L^m)$ for a fixed integer m. Let $S_0, \cdots, S_{d-1}$ be a basis of $H^0(M, L^m)$. The metrics (h, ω_g) define an inner product $(\ ,\ )$ on $H^0(M, L^m)$ as

$$(S_A, S_B)_{h_m} = \int_M < S_A, S_B >_{h_m} dV_g, \quad A, B = 0, \cdots, d-1,$$

where $< S_A, S_B >_{h_m}$ is the pointwise inner product with respect to h_m and $dV_g = \frac{1}{n!}\omega_g^n$.

We assume that at the point $x_0 \in M$,

$$S_0(x_0) \neq 0, \quad S_A(x_0) = 0, \quad A = 1, \cdots, d-1.$$

Suppose

$$F_{AB} = (S_A, S_B)_{h_m}, \quad A, B = 0, \cdots, d-1.$$

Then (F_{AB}) is the metric matrix which is positive Hermitian. Let

$$F_{AB} = \sum_{C=0}^{d-1} G_{AC} \overline{G_{BC}}$$

for a $d \times d$ matrix (G_{AB}) and let (H_{AB}) be the inverse matrix of (G_{AB}). Then it is easy to see that $\{\sum_{B=0}^{d-1} H_{AB} S_B\}, (A = 0, \cdots, d-1)$ forms an orthonormal basis of $H^0(M, L^m)$. By the definition of $S_A (A = 0, \cdots, d-1)$,

we have

(2.1)

$$\sum_A ||\sum_B H_{AB}S_B(x_0)||^2_{h_m} = \sum_A ||H_{A0}S_0(x_0)||^2_{h_m} = \sum_{A=0}^{d-1} |H_{A0}|^2||S_0(x_0)||^2_{h_m}.$$

Suppose (I_{AB}) is the inverse matrix of (F_{AB}). Then by the definition of (H_{AB}), we have

$$\sum_{A=0}^{d-1} |H_{A0}|^2 = I_{00}. \tag{2.2}$$

By the above discussion, we know that in order to prove Theorem 1.1, we just need to estimate the quantity I_{00} and $||S_0||_{h_m}$ at x_0 for a suitable choice of the basis $S_0, \cdots, S_{d-1}$. We use Tian's peak section method in [11] to get the estimates.

We construct peak sections of L^m for m large. Choose a local normal coordinate $(z_1, \cdots, z_n)$ centered at x_0 such that the Hermitian matrix $(g_{\alpha\overline{\beta}})$ satisfies

$$\begin{aligned} &g_{\alpha\overline{\beta}}(x_0) = \delta_{\alpha\beta} \\ &\frac{\partial^{p_1+\cdots+p_n} g_{\alpha\overline{\beta}}}{\partial z_1^{p_1}\cdots\partial z_n^{p_n}}(x_0) = 0 \end{aligned} \tag{2.3}$$

for $\alpha, \beta = 1, \cdots, n$ and any nonnegative integers $p_1, \cdots, p_n$ with $p_1 + \cdots + p_n \neq 0$. Such a local coordinate system exists and is unique up to an affine transformation. This coordinate system is known as the K-coordinate. See [2] or [10] for details.

Next we choose a local holomorphic frame e_L of L at x_0 such that the local representation function a of the Hermitian metric h has the properties

$$a(x_0) = 1, \frac{\partial^{p_1+\cdots+p_n} a}{\partial z_1^{p_1}\cdots\partial z_n^{p_n}}(x_0) = 0 \tag{2.4}$$

for any nonnegative integers $(p_1, \cdots, p_n)$ with $p_1 + \cdots + p_n \neq 0$.

Suppose that the local coordinate $(z_1, \cdots, z_n)$ is defined on an open neighborhood U of x_0 in M. Define the function $|z|$ by $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ for $z \in U$.

Let $\mathbb{Z}_+^n$ be the set of n-tuple of integers $(p_1, \cdots, p_n)$ such that $p_i \geq 0 (i = 1, \cdots, n)$. Let $P = (p_1, \cdots, p_n)$. Define

$$z^P = z_1^{p_1}\cdots z_n^{p_n} \tag{2.5}$$

and

$$p = p_1 + \cdots + p_n.$$

The following lemma is proved in [11] using the standard $\overline{\partial}$-estimates (see e.g. [9]).

Lemma 2.1. *For $P=(p_1,\cdots,p_n)\in\mathbb{Z}_+^n$, and an integer $p'>p=p_1+\cdots+p_n$, there exists an $m_0>0$ such that for $m>m_0$, there is a holomorphic global section $S_{P,m}$ in $H^0(M,L^m)$, satisfying*

$$\int_M ||S_{P,m}||_{h_m}^2 dV_g = 1, \qquad \int_{M\backslash\{|z|\le\frac{\log m}{\sqrt{m}}\}} ||S_{P,m}||_{h_m}^2 dV_g = O(\frac{1}{m^{2p'}}),$$

and $S_{P,m}$ can be decomposed as

$$S_{P,m}=\tilde{S}_{P,m}+u_{P,m}, \qquad (\tilde{S}_{P,m}\text{ and } u_{P,m} \text{ not necessarily continuous})$$

such that

$$\tilde{S}_{P,m}(x)=\begin{cases}\lambda_P z^P e_L^m(1+O(\frac{1}{m^{2p'}})) & x\in\{|z|\le\frac{\log m}{\sqrt{m}}\}\\ 0 & x\in M\backslash\{|z|\le\frac{\log m}{\sqrt{m}}\},\end{cases}$$

$$u_{P,m}(x)=O(|z|^{2p'}) \qquad x\in U,$$

and

$$\int_M ||u_{P,m}||_{h_m}^2 dV_g = O(\frac{1}{m^{2p'}}),$$

where $O(\frac{1}{m^{2p'}})$ denotes a quantity dominated by $C/m^{2p'}$ with the constant C depending only on p' and the geometry of M. Moreover

$$\lambda_P^{-2}=\int_{|z|\le\frac{\log m}{\sqrt{m}}} |z^P|^2 a^m dV_g.$$

□

Because of the above lemma, in the rest of this paper, we will use $S_{P,m}^{p'}$ to denote the peak sections defined above. Furthermore, we always set

$$S_0=S_{(0,\cdots,0),m}^{p'}$$

for p' and m large enough.

We use the notation $|||\cdot|||$ to denote the L^2 norm of a section of $H^0(M,L^m)$. That is, if $T\in H^0(M,L^m)$, then

$$|||T|||=\sqrt{\int_M ||T||_{h_m}^2 dV_g}.$$

The following lemma [10, Lemma 3.2] is a generalization of the lemma of Tian [11, Lemma 2.2].

Lemma 2.2 (Ruan)**.** *Let $S_P=S_{P,m}^{p'}$ be the section constructed in the above lemma. Let T be another section of L^m. Near x_0, $T=fe_L^m$ for a holomorphic function f. When we say T's Taylor expansion at x_0, we mean the Taylor expansion of f at x_0 under the coordinate system $(z_1,\cdots,z_n)$.*

1. *If z^P is not in T's Taylor expansion at x_0, then*

$$(S_P,T)_{h_m}=O(\frac{1}{m})|||T|||.$$

2. *If T contains no terms z^Q, such that $q < p + \sigma$ ($1 \leq \sigma \leq p' - n - p$, and σ is an integer. Recall that $p = p_1 + \cdots + p_n$ and $q = q_1 + \cdots + q_n$.) in the Taylor expansion at x_0, then*

$$(S_P, T)_{h_m} = O(\frac{1}{m^{1+\sigma/2}})|||T|||. \tag{2.6}$$

Proof: We only prove 2, since 1 is similar and easier.

By Lemma 2.1, we have the decomposition

$$S^{p'}_{P,m} = \tilde{S}_{P,m} + u_{P,m} = \tilde{S}_P + u_P$$

with $|||u_P|||^2 = O(\frac{1}{m^{2p'}})$. Thus

$$(u_P, T)_{h_m} = O(\frac{1}{m^{p'}})|||T||| = O(\frac{1}{m^{1+\sigma/2}})|||T|||$$

by the Cauchy inequality. For the $\tilde{S}_P$ part, we have

$$\begin{aligned}
&\int_{|z| \leq \frac{\log m}{\sqrt{m}}} < \tilde{S}_P, T >_{h_m} dV_g \\
&\qquad = \lambda_P \int_{|z| \leq \frac{\log m}{\sqrt{m}}} < \tilde{S}'_P, T >_{h_m} dV_g + O(\frac{1}{m^{2p'}})\lambda_P |||T|||,
\end{aligned}$$

where $\tilde{S}'_P = z^P e_L^m$ for $|z| \leq \frac{\log m}{\sqrt{m}}$ and is zero otherwise.

Let dV_0 be the Euclidean volume. i.e.,

$$dV_0 = (\frac{\sqrt{-1}}{2\pi})^n dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n.$$

We have

$$dV_g \geq c dV_0, \qquad a^m \geq c e^{-m|z|^2}$$

on U for a suitable constant $c > 0$ and for m large. Thus we have

$$\int_{|z| \leq \frac{\log m}{\sqrt{m}}} |z^P|^2 a^m dV_g \geq c^2 \int_{|z| \leq \frac{\log m}{\sqrt{m}}} |z^P|^2 e^{-m|z|^2} dV_0.$$

By the simple combinatorial identity

$$\int_{\mathbb{C}^n} |z^P|^2 e^{-m|z|^2} dV_0 = \frac{P!}{m^{n+p}}, \tag{2.7}$$

where $P! = p_1! \cdots p_n!$, we see that

$$\int_{|z| \leq \frac{\log m}{\sqrt{m}}} |z^P|^2 a^m dV_g \geq c_1 \frac{1}{m^{n+p}}$$

for some number $c_1 > 0$ independent to m. Thus by the definition of λ_P,

$$\lambda_P \leq \frac{1}{\sqrt{c_1}} m^{(n+p)/2}. \tag{2.8}$$

In order to prove (2.6), we then just need to prove

$$\lambda_P \int_{|z|\leq \frac{\log m}{\sqrt{m}}} < \tilde{S}'_P, T >_{h_m} dV_g = O(\frac{1}{m^{1+\frac{\sigma}{2}}})|||T|||.$$

Let

$$\xi(x) = \log a + |z|^2 \qquad (2.9)$$
$$\eta(x) = \log\det g_{\alpha\overline{\beta}}.$$

If $|z| \leq \frac{\log m}{\sqrt{m}}$, then $|m\xi| \leq \frac{C}{\sqrt{m}}$. Since

$$|e^{m\xi} - 1 - m\xi - \cdots - \frac{1}{(\sigma+4)!} m^{\sigma+4}\xi^{\sigma+4}| \leq m^{\sigma+5}|\xi|^{\sigma+5}e^{m|\xi|},$$

we have

$$\begin{aligned}
&\lambda_P | \int_{|z|\leq \frac{\log m}{\sqrt{m}}} e^{-m\xi+m|z|^2} < \tilde{S}'_P, T >_{h_m} (e^{m\xi} - 1 - m\xi \\
&\quad - \cdots - \frac{1}{(\sigma+4)!} m^{\sigma+4}\xi^{\sigma+4}) e^{-m|z|^2} dV_g| \\
&\leq C\lambda_P \int_{|z|\leq \frac{\log m}{\sqrt{m}}} | < \tilde{S}'_P, T >_{h_m} | m^{\sigma+5}|\xi|^{\sigma+5} dV_g \\
&\leq C\lambda_P \sqrt{\int_{|z|\leq \frac{\log m}{\sqrt{m}}} |z^P|^2 m^{2\sigma+10}|\xi|^{2\sigma+10} e^{-m|z|^2} dV_g} |||T|||,
\end{aligned} \qquad (2.10)$$

where C is a constant independent to m. Using $|m\xi| \leq \frac{C}{\sqrt{m}}$ again, we have

$$\lambda_P \sqrt{\int_{|z|\leq \frac{\log m}{\sqrt{m}}} |z^P|^2 m^{2\sigma+10}|\xi|^{2\sigma+10} e^{-m|z|^2} dV_g} = O(\frac{1}{m^{\sigma+5}}) = O(\frac{1}{m^{1+\sigma/2}}).$$

Thus in order to prove the lemma, we need only to prove that for any $k \leq \sigma + 4$,

$$\lambda_P \int_{|z|\leq \frac{\log m}{\sqrt{m}}} e^{-m\xi+m|z|^2} < \tilde{S}'_P, T >_{h_m} m^k \xi^k e^{-m|z|^2} dV_g = O(\frac{1}{m^{1+\frac{\sigma}{2}}})|||T|||.$$

Let $\xi = \xi_1 + \xi_2$ be the decomposition of ξ such that ξ_1 contains those terms of order less than or equal to $4\sigma + 12$ and ξ_2 contains those terms of order greater than $4\sigma + 12$. Using the similar method as above, we can proved that

$$\lambda_P \int_{|z|\leq \frac{\log m}{\sqrt{m}}} < \tilde{S}'_P, T >_{h_m} m^k((\xi_1+\xi_2)^k - \xi_1^k) e^{-m\xi} dV_g = O(\frac{1}{m^{1+\frac{\sigma}{2}}})|||T|||$$

because $|(\xi_1 + \xi_2)^k - \xi_1^k| \leq C/m^{2\sigma+5}$. Thus we only need to prove that

$$\lambda_P \int_{|z|\leq \frac{\log m}{\sqrt{m}}} < \tilde{S}'_P, T >_{h_m} m^k \xi_1^k e^{-m\xi} dV_g = O(\frac{1}{m^{1+\frac{\sigma}{2}}})|||T|||$$

for $k \le \sigma + 4$. Let

$$e^{\eta} = \eta_1 + \eta_2,$$

where η_1 consists of the terms of order less than or equal to $4\sigma + 12$ and $\eta_2 = e^{\eta} - \eta_1$. Then as above

$$\lambda_P \int_{|z| \le \frac{\log m}{\sqrt{m}}} < \tilde{S}'_P, T >_{h_m} m^k \xi_1^k \eta_2 e^{-m\xi} dV_0 = O(\frac{1}{m^{1+\frac{\sigma}{2}}})|||T|||,$$

where dV_0 is the Euclidean volume form.

It remains to prove that

$$\lambda_P \int_{|z| \le \frac{\log m}{\sqrt{m}}} < \tilde{S}'_P, T >_{h_m} m^k \xi_1^k \eta_1 e^{-m\xi} dV_0 = O(\frac{1}{m^{1+\frac{\sigma}{2}}})|||T|||.$$

Note that $\xi_1^k \eta_1$ is a polynomial of z and $\overline{z}$ whose number of terms is bounded by a constant only depending on σ and n. Let

$$\xi_1^k \eta_1 = \sum \xi_{IJ} z^I \overline{z}^J. \tag{2.11}$$

If $|I| - |J| < \sigma$, then by the assumption on T,

$$\int_{|z| \le \frac{\log m}{\sqrt{m}}} < \tilde{S}'_P, T >_{h_m} m^k \xi_{IJ} z^I \overline{z}^J e^{-m\xi} dV_0 = 0.$$

On the other hand, under the K-coordinates, in the expansion of ξ, there are no z^P or $z^P \overline{z}$ terms. Thus in (2.11), we must have $|J| \ge 2k$. If $|I| - |J| \ge \sigma \ge 1$, then $|J| \ge 1$. So

$$|I| + |J| - 2k \ge \sigma + 2.$$

Thus

$$\begin{aligned} &\lambda_P \int_{|z| \le \frac{\log m}{\sqrt{m}}} < \tilde{S}'_P, T >_{h_m} m^k \xi_1^k \eta_1 e^{-m\xi} dV_0 \\ &\le C \lambda_P \sqrt{\int_{|z| \le \frac{\log m}{\sqrt{m}}} |z^P|^2 m^{2k} |z|^{2(2k+\sigma+2)} e^{-m|z|^2} dV_0} |||T|||. \end{aligned}$$

The lemma follows from (2.8) and the elementary fact that

$$\int_{\mathbb{C}^n} |z^P|^2 |z|^{2q} e^{-m|z|^2} dV_0 = \frac{(n+p+q-1)! P!}{(p+n-1)! m^{n+p+q}}. \tag{2.12}$$

□

In the above lemma, if T itself is a peak section, then we have a more accurate result. Before going to the result, let's first define the weight and the index of a polynomial (resp. monomial, series).

Definition 2.1. *Let R be a component of the i-th order covariant derivative of the curvature tensor, or the Ricci tensor, or the scalar curvature at a fixed point where $i \ge 0$. Define the weight $w(R)$ of R to be the number $(1 + \frac{i}{2})$. For example,*

$$w(R_{i\bar{j}k\bar{l}}) = w(R_{i\bar{j}}) = w(\rho) = 1$$

and

$$w(R_{i\bar{j}k\bar{l},m}) = \frac{3}{2}.$$

The above definition of the weight can be naturally extended to any monomial of components of the curvature tensor and its derivatives. If a polynomial (resp. series) whose any term is of weight i, then we call the polynomial (resp. series) is of weight i.

More generally, we have the following definition of index.

Definition 2.2. *Let f be a monomial of the form*

$$m^\mu g z^P \overline{z}^Q,$$

where μ is an integer or half of an integer, P, Q are multiple index, z^P and z^Q are defined as in (2.5), and g is a monomial of components of the curvature tensor and its derivatives at a fixed point. In the rest of this paper, a polynomial (resp. series) is always a polynomial generated by the monomials of the above form. Define the index of f by

$$ind(f) = \mu + w(g) - \frac{p+q}{2},$$

where $p = p_1 + \cdots + p_n$ and $q = q_1 + \cdots + q_n$. We say a polynomial (resp. series) is of index i, if all of its monomials have the same index i. In that case, we say that the polynomial (resp. series) is homogeneous. The polynomials (resp. series) of index 0 is called regular.

It is easy to check that if f_1, f_2 are homogeneous polynomials (resp. series), then

$$ind(f_1 f_2) = ind(f_1) + ind(f_2).$$

The regular polynomials (resp. series) form a ring under the addition and multiplication.

When a polynomial (resp. monomial, series) contains no m or z's, the index is the same as the weight. The following two lemmas give the motivation of the above definitions.

Lemma 2.3. *The Taylor expansion of the function*

$$a^m \det(g_{\alpha\overline{\beta}}) e^{m|z|^2}.$$

at x_0 is a regular series.

Proof: Consider the Taylor expansion of ξ and η in (2.9) under the K-coordinates. It is not hard to see that the Taylor expansion of ξ is of index (-1) and the Taylor expansion of η is regular. Thus the Taylor expansion of $m\xi + \eta$ is regular and so is the Taylor expansion of

$$a^m \det g_{\alpha\overline{\beta}} e^{m|z|^2} = e^{m\xi+\eta}.$$

$\square$

Lemma 2.4. *If*

$$A_1 + \cdots + A_t$$

is a polynomial of index i, then there is a polynomial B of index $i-n$ such that for any $p'>0$,

$$\int_{|z|\leq\frac{\log m}{\sqrt{m}}} A_1 e^{-m|z|^2} dV_0 + \cdots + \int_{|z|\leq\frac{\log m}{\sqrt{m}}} A_t e^{-m|z|^2} dV_0 - B = O(\frac{1}{m^{p'}}).$$

Proof: Suppose that

$$A_k = m^\mu g z^P \overline{z}^Q, \quad 1\leq k\leq t.$$

If $P\neq Q$, then

$$\int_{|z|\leq\frac{\log m}{\sqrt{m}}} A_k e^{-m|z|^2} dV_0 = 0.$$

If $P=Q$, then by (2.12),

$$\int_{|z|\leq\frac{\log m}{\sqrt{m}}} A_k e^{-m|z|^2} dV_0 = Cm^{\mu-p-n} g + O(\frac{1}{m^{p'}})$$

for any $p'>0$ where C is a constant. Since A_k is of index i, we have

$$\mu + w(g) - p = i.$$

Thus

$$ind(Cm^{\mu-p-n}g) = \mu - p - n + w(g) = i - n.$$

The lemma is proved.

□

Proposition 2.1. *We have the following expansion for any $p' > t + 2(n+p+q)$,*

$$(S^{p'}_{P,m}, S^{p'}_{Q,m}) = \frac{1}{m^\delta}(a_0 + \frac{a_1}{m} + \cdots + \frac{a_{t-1}}{m^{t-1}} + O(\frac{1}{m^t})),$$

where $\delta = 1$ or $1/2$ and where all the a_i's are polynomials of the curvature and its derivatives such that

$$ind(a_i) = i + \delta.$$

In particular, the series is regular. Moreover, all $a_i, (1\leq i\leq t-1)$ can be found by finite steps of algebraic operations from the curvature and its derivatives.

Proof: The expansion of the function $z^P\overline{z}^Q e^{m\xi+\eta}$ has index $-\frac{p+q}{2}$ by Lemma 2.3. Thus by Lemma 2.4, there is a polynomial B_{PQ} of index $(-\frac{p+q}{2} - n)$ of the form

$$B_{PQ} = s_0 + \frac{s_1}{m} + \cdots + \frac{s_{t-1}}{m^{t-1}}$$

such that

$$\int_{|z|\leq\frac{\log m}{\sqrt{m}}} z^P\overline{z}^Q e^{m\xi+\eta}e^{-m|z|^2}dV_0 = B_{PQ}+O(\frac{1}{m^{p'}}).$$

In particular, $m^{n+p}B_{PP}$ will be regular. Furthermore,

$$B_{PP}\sim\frac{C_P}{m^{n+p}}$$

for constant $C_P\neq 0$ by the same argument as in Equation (2.8). Thus the expansion of

$$m^{\frac{n+p}{2}}\sqrt{B_{PP}}$$

is regular and $\frac{1}{m^{\frac{n+p}{2}}\sqrt{B_{PP}}}$ expands as a regular series. The lemma follows from Lemma 2.2 and the fact that

$$(S^{p'}_{P,m},S^{p'}_{Q,m})=\frac{B_{PQ}}{\sqrt{B_{PP}}\sqrt{B_{QQ}}}+O(\frac{1}{m^{n+p+q+t}}).$$

□

3. The Iteration Process

The main result of this section is Theorem 3.1. In order to obtain the result, we basically use the iteration process in the numerical analysis for finding the inverse matrix of a given tri-diagonal matrix.

Theorem 3.1. *Let $x_0\in M$. Suppose $\{S_0,S_1,\cdots,S_{d-1}\}$ is a basis of $H^0(M,L^m)$ with $S_0(x_0)\neq 0$ and $S_A(x_0)=0$ for $A=1,\cdots,d-1$. Let $(F_{AB})=((S_A,S_B)_{h_m})(A,B=0,\cdots,d-1)$ be the metric matrix. Let (I_{AB}) be the inverse matrix of (F_{AB}). Then for any positive integer $p>0$, we have the expansion*

$$I_{00}=1+\frac{\sigma_3}{m^3}+\frac{\sigma_{7/2}}{m^{7/2}}+\cdots+\frac{\sigma_{p-1}}{m^{p-1}}+\frac{\sigma_{((2p-1)/2)}}{m^{(2p-1)/2}}+O(\frac{1}{m^p}).$$

Furthermore, $\sigma_k(k=3,7/2,4,\cdots,(2p-1)/2)$ are polynomials of weight k of the curvature and the derivatives of the curvature at x_0.

Remark 3.1. Although not needed, a more careful analysis will show that $\sigma_{(k/2)}=0$ for all odd k's.

Before proving the theorem, we need some algebraic preparation.

Definition 3.1. *We say $M=\{M(m)\}$ is a sequence of $s\times s$ block matrices with block number $t\in\mathbb{Z}$, if for each m,*

$$M=M(m)=\begin{pmatrix}M_{11}(m)&\cdots&M_{1t}(m)\\ \vdots&\ddots&\vdots\\ M_{t1}(m)&\cdots&M_{tt}(m)\end{pmatrix}$$

such that for $1 \leq i, j \leq t$, M_{ij} is a $\sigma(i) \times \sigma(j)$ matrix and

$$\sum_{i=1}^{t} \sigma(i) = s,$$

where $\sigma : \{1, \cdots, t\} \to \mathbb{Z}_+$ assigns each number in $\{1, \cdots, t\}$ a positive integer. We say that $\{M(m)\}$ is of type $A(p)$ for a positive integer p, if for any entry s of the matrix M ,we have

1. *If s is a diagonal entry of $M_{ii}(1 \leq i \leq t)$, then we have the the following Taylor expansion*
$$s = 1 + \frac{s_1}{m} + \cdots + \frac{s_{p-1}}{m^{p-1}} + O(\frac{1}{m^p}).$$
2. *If s is not a diagonal entry of $M_{ii}(1 \leq i \leq t)$, then we have the Taylor expansion*
$$s = \frac{1}{m^\delta}(s_0 + \frac{s_1}{m} + \cdots + \frac{s_{p-1}}{m^{p-1}} + O(\frac{1}{m^p})),$$
where δ is equal to 1 or $\frac{3}{2}$;
3. *If s is an entry of the matrix M_{ij} for which $|i-j| = 1$, then $s = O(\frac{1}{m^{\frac{3}{2}}})$. In addition, if $i \neq t$ or $j \neq t$, then we have the Taylor expansion*
$$s = \frac{1}{m^\delta}(s_0 + \frac{s_1}{m} + \cdots + \frac{s_{p-1}}{m^{p-1}} + O(\frac{1}{m^p})),$$
where δ is equal to 1 or $\frac{3}{2}$;
4. *If s is an entry of M_{ij} for which $|i-j| > 1$, then*
$$s = O(\frac{1}{m^p}).$$

The set of all quantities $(s_1/m, \cdots, s_{p-1}/m^{p-1})$, or $(\frac{s_0}{m^\delta}, \frac{s_1}{m^{1+\delta}}, \cdots, \frac{s_{p-1}}{m^{p+\delta-1}})$ for s running from all the entries of M_{ij} where $|i-j| \leq 1$ and $i \neq t$ or $j \neq t$ are called the Taylor Data of order p.

Remark 3.2. Since $M_{ij} = O(\frac{1}{m^p})$ for $|i-j| > 1$, it can be treated as zero when we are only interested in the expansion of order up to $p-1$. A matrix whose entries $M_{ij} = 0$ for $|i-j| > 1$ is called a tri-diagonal matrix. For such a matrix, we have a simple iteration process for finding its inverse matrix.

The following proposition is a modification of the iteration process in the numerical analysis for finding the inverse matrix of a given tri-diagonal matrix.

Proposition 3.1. *Let $M = M(m)$ be a sequence of $s \times s$ block matrices with block number $t = p+1$ and be of type $A(p)$. We further assume that $M = M(m)$ is Hermitian positive. Let*

$$N = N(m) = \begin{pmatrix} N_{11} & \cdots & N_{1t} \\ \vdots & \ddots & \vdots \\ N_{t1} & \cdots & N_{tt} \end{pmatrix}$$

be the inverse block matrix of $M(m)$. By the inverse block matrix, we mean the inverse matrix of the original matrix with the same block partition as that of the original block matrix. We have the following asymptotic expansion:

(3.1)

$$N_{11} = N_{11}^{(0)} + \frac{N_{11}^{(1/2)}}{m^{1/2}} + \frac{N_{11}^{(1)}}{m} + \cdots + \frac{N_{11}^{(p-1)}}{m^{p-1}} + \frac{N_{11}^{((2p-1)/2)}}{m^{(2p-1)/2}} + O(\frac{1}{m^p}).$$

Furthermore, all entries of $N_{11}^{\alpha}/m^{\alpha}, (0 \leq \alpha \leq (2p-1)/2)$ are polynomials of Taylor Data of order p of $M = M(m)$.

We need the following elementary lemma.

Lemma 3.1. *Suppose*

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

is an invertible block matrix which is Hermitian positive. $\overline{M_{11}^T} = M_{11}$, $\overline{M_{22}^T} = M_{22}$ and $\overline{M_{12}^T} = M_{21}$. Let

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$$

be the inverse block matrix of M. Then

$$N_{11} = M_{11}^{-1} + M_{11}^{-1} M_{12} (M_{22} - M_{21} M_{11}^{-1} M_{12})^{-1} M_{21} M_{11}^{-1}.$$

The proof of the lemma is elementary and is omitted.

□

Proof of the Proposition: Suppose $p = 1$. Then by Lemma 3.1, we have

$$N_{11} = M_{11}^{-1} + M_{11}^{-1} M_{12} (M_{22} - M_{21} M_{11}^{-1} M_{12})^{-1} M_{21} M_{11}^{-1}.$$

Since $M_{12} = O(\frac{1}{m})$, we see that

$$N_{11} = M_{11}^{-1} + O(\frac{1}{m}) = E(\sigma(1)) + O(\frac{1}{m}),$$

where $E(\sigma(1))$ is the $\sigma(1) \times \sigma(1)$ unit matrix. Thus the proposition is true in the case $p = 1$.

Assuming that when $p = k$, the proposition is true. Let $p = k + 1$. Using Lemma 3.1, we have

$$N_{11} = M_{11}^{-1} + M_{11}^{-1} (M_{12}\, M_{13}\, \cdots\, M_{1(k+2)}) \tilde{M}^{-1} (M_{21}\, \cdots\, M_{(k+2)1})^T M_{11}^{-1},$$

where by Lemma 3.1,

$$\tilde{M} = \begin{pmatrix} M_{22} & \cdots & M_{2(k+2)} \\ \vdots & \ddots & \vdots \\ M_{(k+2)2} & \cdots & M_{(k+2)(k+2)} \end{pmatrix} - \begin{pmatrix} M_{21} \\ \vdots \\ M_{(k+2)1} \end{pmatrix} M_{11}^{-1} \begin{pmatrix} M_{12} & \cdots & M_{1(k+2)} \end{pmatrix}.$$

By the assumption, $M = M(m)$ is a sequence of matrices with the block number $k+2$. Since $M_{r1} = O(\frac{1}{m^{k+1}})(r > 2)$, it is easy to see that $\tilde{M} = \tilde{M}(m)$ is also a sequence of block matrix with the block number $k+1$ and is of type $A(k)$. Furthermore, we have

$$N_{11} = M_{11}^{-1} + M_{11}^{-1} M_{12} \tilde{N}'_{22} M_{21} M_{11}^{-1} + O(\frac{1}{m^{k+1}}), \tag{3.2}$$

where

$$\tilde{N} = \begin{pmatrix} \tilde{N}'_{22} & \cdots & \tilde{N}'_{2(k+2)} \\ \vdots & \ddots & \vdots \\ \tilde{N}'_{(k+2)2} & \cdots & \tilde{N}'_{(k+2)(k+2)} \end{pmatrix}$$

is the inverse matrix of $\tilde{M}$. Then by the induction assumption, we have

$$\tilde{N}'_{22} = \tilde{N}'^{(0)}_{22} + \frac{\tilde{N}'^{(1/2)}_{22}}{m^{1/2}} + \frac{\tilde{N}'^{(1)}_{22}}{m} + \cdots + \frac{\tilde{N}'^{(k-1)}_{22}}{m^{k-1}} + \frac{\tilde{N}'^{((2k-1)/2)}_{22}}{m^{(2k-1)/2}} + O(\frac{1}{m^k}),$$

where $\tilde{N}'^{(i/2)}_{22}/m^{i/2}(1 \le i \le (2k-1)/2)$ are polynomials of Taylor Data of order k of $\tilde{M} = \tilde{M}(m)$. Thus $\tilde{N}'^{(i/2)}_{22}/m^{i/2}(1 \le i \le (2k-1)/2)$ must be polynomials of Taylor Data of order $(k+1)$ of $M = M(m)$. On the other hand, all the entries of the matrix M_{11}^{-1} are polynomials of Taylor Data of order $(k+1)$ of $M = M(m)$. Since $M_{12} = O(\frac{1}{m})$, the entries of N_{11} are polynomials of Taylor Data of order $(k+1)$ of $M = M(m)$ by (3.2). The proposition is proved.

Proof of Theorem 3.1: For a multiple indices, define $|P| = p_1 + \cdots + p_n$. Suppose

$$V_k = \{S \in H^0(M, L^m) | D^Q S(x_0) = 0 \,\text{for}\, |Q| \le k\}$$

for $k = 1, 2, \cdots$, where $Q \in \mathbb{Z}^n_+$ is a multiple indices, and D is a covariant derivative on the bundle L^m. $V_k = \{0\}$ for k sufficiently large. For fixed p, let $p' = n + 8p(p-1)$. Suppose that m is large enough such that $H^0(M, L^m)$ is spanned by the $S^{p'}_{P,m}$'s for the multiple indices $|P| \le 2p(p-1)$ and $V_{2p(p-1)}$. Let $r = d - \dim V_{2p(p-1)}$. Then r only depends on p and n. Let $T_1, \cdots, T_{d-r}$ be an orthnormal basis of $V_{2p(p-1)}$ such that

$$(S^{p'}_{P,m}, T_\alpha) = 0$$

for $|P| \leq 2p(p-1)$ and $\alpha > r$. Let $s(k) = \dim V_k$ for $k \in \mathbb{Z}$. For any $1 \leq i, j \leq p$, let M_{ij} be the matrix formed by $(S^{p'}_{P,m}, S^{p'}_{Q,m})$ where $2p(i-2) \leq |P| \leq 2p(i-1)$ and $2p(j-2) \leq |Q| \leq 2p(j-1)$. Furthermore, define $M_{i(p+1)}$ to be the matrix whose entries are (S_P, T_α) for $2p(i-2) \leq |P| \leq 2p(i-1)$ and $1 \leq \alpha \leq r$. Define $M_{(p+1)i}$ to be the complex conjugate of $M_{i(p+1)}$. Finally, define $M_{(p+1)(p+1)}$ to be the $r \times r$ unit matrix $E(r)$. Then it is easy to check that $M = (M_{ij})$ is a sequence of block matrices of type $A(p)$ with the block number $p+1$ by using Lemma 2.2 and Proposition 2.1.

Define an order $\geq$ on the multiple indices P as follows: $P \geq Q$, if

1. $|P| > |Q|$ or;
2. $|P| = |Q|$ and $p_j = q_j$ but $p_{j+1} > q_{j+1}$ for some $0 \leq j \leq n$.

Using this order, there is a one-one order preserving correspondence κ between $\{0, \cdots, r-1\}$ and $\{P \| P| \leq 2p(p-1)\}$.

Define

$$S_A = \begin{cases} S^{p'}_{\kappa(A),m} & A \leq r-1 \\ T_{A-r+1} & A \geq r \end{cases}.$$

Comparing the matrix M to the metric matrix $F_{AB} = ((S_A, S_B)), (A, B = 0, \cdots, d-1)$, by the choice of the basis, we see that

$$(F_{AB}) = \begin{pmatrix} M & 0 \\ 0 & E(d-2r) \end{pmatrix},$$

where $E(d-2r)$ is the $(d-2r) \times (d-2r)$ identity matrix. If $N = (N_{ij})$ is the inverse matrix of M, then N_{11} is an 1×1 matrix and

$$I_{00} = N_{11}.$$

So Proposition 3.1 gives the desired asymptotic expansion

$$I_{00} = \sigma_0 + \frac{\sigma_{1/2}}{m^{1/2}} + \frac{\sigma_1}{m} + \cdots + \frac{\sigma_{p-1}}{m^{p-1}} + \frac{\sigma_{((2p-1)/2)}}{m^{(2p-1)/2}} + O(\frac{1}{m^p}).$$

Moreover, Proposition 3.1 states that $\sigma_k/m^k, (k = 1/2, \cdots, (2p-1)/2)$ are polynomials of the Taylor Data of order p of M. By Proposition 2.1, the Taylor Data for the inner products are regular. Thus $\sigma_k/m^k, (1/2 \leq k \leq (2p-1)/2)$ are regular or in other word, σ_k is a polynomial of the curvature and its derivatives of weight k for $k = 1/2, \cdots, (2p-1)/2$.

It remains to show that

$$\sigma_0 = 1, \, \sigma_{1/2} = \sigma_1 = \sigma_{3/2} = \sigma_2 = \sigma_{5/2} = 0.$$

This can be seen using the following argument. Let $S_0, S_1, \cdots, S_{d-1}$ be a basis of $H^0(M, L^m)$. We suppose that $S_A(x_0) = 0$ for $A = 1, \cdots, d-1$. We also assume that $(S_0, S_A) = 0$ for $A > 1$. Let $c = (S_0, S_1)$. Then $I_{00} = (1-|c|^2)^{-1}$. By Lemma 2.2, we see that $c = O(\frac{1}{m^{3/2}})$. Thus $I_{00} = 1 + O(\frac{1}{m^3})$.

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 except that we postpone the computation of the a_3 term to the next section. The method of computing the a_3 term is the same as that of a_j for $j = 0, 1, 2$. We put it off to the next section due to the complexity in the computation.

By Theorem 3.1 and Equation (2.1), we just need to estimate

$$|\lambda_{(0,\cdots,0)}|^{-2} = \int_{|z|\leq\frac{\log m}{\sqrt{m}}} a^m dV_g \tag{4.1}$$

to the term $\frac{1}{m^{n+2}}$, from which the first three coefficients a_0, a_1 and a_2 can be calculated.

First we define our notations in the following equations (4.2)- (4.6).

The curvature tensor is defined as

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - \sum_{p=1}^n \sum_{q=1}^n g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l} \tag{4.2}$$

for $i, j, k, l = 1, \cdots, n$. The Ricci curvature is

$$R_{i\bar{j}} = -\sum_{k,l=1}^n g^{k\bar{l}} R_{i\bar{j}k\bar{l}} \tag{4.3}$$

for $i, j = 1, \cdots, n$, and the scalar curvature is the trace of the Ricci curvature

$$\rho = \sum_{i,j=1}^n g^{i\bar{j}} R_{i\bar{j}}. \tag{4.4}$$

The covariant derivative with respect to $\frac{\partial}{\partial z_p}$ of the curvature tensor is defined as

$$R_{i\bar{j}k\bar{l},p} = \frac{\partial}{\partial z_p} R_{i\bar{j}k\bar{l}} - \sum_{s=1}^n \Gamma^s_{ip} R_{s\bar{j}k\bar{l}} - \sum_{s=1}^n \Gamma^s_{kp} R_{i\bar{j}s\bar{p}}, \tag{4.5}$$

where $\Gamma^k_{ij} = \sum_{s=1}^n g^{k\bar{s}} \frac{\partial g_{i\bar{s}}}{\partial z_j}$ is the Christoffel symbol. Higher derivatives are defined in the similar way. Finally, the Laplace operator is denoted by Δ:

$$\Delta = \sum_{i=1}^n \sum_{j=1}^n g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}. \tag{4.6}$$

As will be made obvious, in order to estimate $|\lambda_{(0,\cdots,0)}|^{-2}$ up to the term $\frac{1}{m^{n+2}}$, we must use the Taylor expansion of $\log a$ up to degree 6 and the Taylor expansion of $\log \det g_{\alpha\bar{\beta}}$ up to degree 4. Suppose we have the following expansions at x_0:

$$\begin{aligned} \log a &= -|z|^2 + e_4 + e_5 + e_6 + O(|z|^7) \\ \log\det(g_{\alpha\bar{\beta}}) &= c_2 + c_3 + c_4 + O(|z|^5), \end{aligned} \tag{4.7}$$

where e_4, e_5 and e_6 are homogeneous polynomials of z and $\overline{z}$ of degree 4,5 and 6, respectively and c_2, c_3, and c_4 are homogeneous polynomials of degree 2,3 and 4, respectively. Then a straightforward computation gives

$$\tag{4.8} \begin{cases} e_4 = -\frac{1}{4}R_{i\bar{j}k\bar{l}}z_iz_k\overline{z}_j\overline{z}_l \\ e_5 = -\frac{1}{12}R_{i\bar{j}k\bar{l},p}z_iz_kz_p\overline{z}_j\overline{z}_l - \frac{1}{12}R_{i\bar{j}k\bar{l},\bar{q}}z_iz_k\overline{z}_j\overline{z}_l\overline{z}_q \\ \tilde{e}_6 = -\frac{1}{36}(R_{i\bar{j}k\bar{l},p\bar{q}} + R_{i\bar{s}p\bar{q}}R_{s\bar{j}k\bar{l}} + R_{k\bar{s}p\bar{q}}R_{s\bar{l}i\bar{j}} \\ \qquad +R_{i\bar{s}k\bar{q}}R_{s\bar{j}p\bar{l}})z_iz_kz_p\overline{z}_j\overline{z}_l\overline{z}_q \end{cases},$$

where $\tilde{e}_6$ is the $(3,3)$ part of e_6, i.e.,

$$\tilde{e}_6 = \frac{1}{36}\frac{\partial^6 \log a}{\partial z_i\partial z_k\partial z_p\partial\overline{z}_j\partial\overline{z}_l\partial\overline{z}_q}z_iz_kz_p\overline{z}_j\overline{z}_l\overline{z}_q,$$

and

$$\tag{4.9} \begin{cases} c_2 = -R_{i\bar{j}}z_i\overline{z}_j \\ c_3 = -\frac{1}{2}R_{i\bar{j},k}z_iz_k\overline{z}_j - \frac{1}{2}R_{i\bar{j},\bar{l}}z_i\overline{z}_j\overline{z}_l \\ \tilde{c}_4 = -\frac{1}{4}(R_{i\bar{j},k\bar{l}} + R_{i\bar{s}k\bar{l}}R_{s\bar{j}})z_iz_k\overline{z}_j\overline{z}_l \end{cases},$$

where $\tilde{c}_4$ is the $(2,2)$ part of c_4. Here $R_{i\bar{j}k\bar{l}}$, etc denote the value $R_{i\bar{j}k\bar{l}}(x_0)$.

Considering the function $e^{m(\log a+|z|^2)}e^{\log\det g_{\alpha\overline{\beta}}}$, by (4.7), we have

$$\begin{aligned} &e^{m(\log a+|z|^2)}e^{\log\det g_{i\bar{j}}} = e^{m(e_4+e_5+e_6+O(|z|^7))}e^{c_2+c_3+c_4+O(|z|^5)} \\ &= (1+m(e_4+e_5+e_6)+\frac{1}{2}m^2(e_4^2)+O(\cdots)) \\ &\qquad (1+c_2+c_3+c_4+\frac{1}{2}(c_2^2)+O(|z|^5)) \\ &= 1+m(e_4+e_5+e_6)+\frac{1}{2}m^2(e_4^2) \\ &\quad + c_2+mc_2e_4+c_4+\frac{1}{2}(c_2^2)+O(\cdots)), \end{aligned}$$

where $O(\cdots)$ represents the sum of terms, each of which is less than a constant multiple of $m^\mu|z|^t$ for some μ and t such that $t-2\mu>4$. Those terms will not affect the value of a_i, $i=0,1,2$ and can be omitted.

Let φ be a function on a neighborhood of the original point of $\mathbb{C}^n$. For large m, define

$$\tag{4.10} K(\varphi) = \int_{|z|\le\frac{\log m}{\sqrt{m}}}\varphi e^{-m|z|^2}dV_0.$$

Since the functional K is an integration against a symmetric domain, we have $K(e_5)=0$. Thus

$$\tag{4.11} \begin{aligned} &|\lambda_{(0,\cdots,0)}|^{-2} = K(e^{m(|z|^2+\log a)}e^{\log\det g_{i\bar{j}}}) \\ &= K(1)+mK(e_4)+mK(e_6)+\frac{1}{2}m^2K(e_4^2) \\ &+ K(c_2)+mK(c_2e_4)+K(c_4)+\frac{1}{2}K(c_2^2)+O(\frac{1}{m^{5/2}}). \end{aligned}$$

We need the following combinatorial lemma and its corollary which greatly simplified our computation in this paper. In fact, it makes our computation feasible.

Suppose $t > 0$ is an integer. A function A on $\{1, \cdots, n\}^p \times \{1, \cdots, n\}^p$ is called symmetric, if

$$A(\sigma(I), \eta(J)) = A(I, J)$$

where $I, J \in \{1, \cdots, n\}^p$ and $\sigma, \eta \in \Sigma$, the transformation group of $\{1, \cdots, n\}$.

For the sake of simplicity, we set $A_{I,\overline{J}} = A(I, J)$.

Lemma 4.1. *Let A be a symmetric function on $\{1, \cdots, n\}^p \times \{1, \cdots, n\}^p$. Then for any $p' > 0$,*

$$\begin{aligned}
&\sum_{I,J} \int_{|z| \le \frac{\log m}{\sqrt{m}}} A_{I,\overline{J}} z_{i_1} \cdots z_{i_p} \overline{z_{j_1} \cdots z_{j_p}} |z|^{2q} e^{-m|z|^2} dV_0 \\
&= (\sum_I A_{I,\overline{I}}) \frac{p!(n+p+q-1)!}{(p+n-1)! m^{n+p+q}} + O(\frac{1}{m^{p'}}),
\end{aligned}$$

where $I = (i_1, \cdots, i_p)$, $J = (j_1, \cdots, j_p)$ and $1 \le i_1, \cdots, i_p, j_1, \cdots, j_p \le n$.

Proof: Fixing $I = (i_1, \cdots, i_p)$, suppose

$$z_{i_1} \cdots z_{i_p} = z_1^{p_1} \cdots z_n^{p_n}.$$

Then

$$\sum_{i=1}^n p_i = p.$$

It is easy to see that if $\sigma(J) \ne I$ for any $\sigma \in \Sigma$, then

$$\int_{|z| \le \frac{\log m}{\sqrt{m}}} z_{i_1} \cdots z_{i_p} \overline{z_{j_1} \cdots z_{j_p}} |z|^{2q} e^{-m|z|^2} dV_0 = 0.$$

On the other hand, if $\sigma(I) = J$ for some $\sigma \in \Sigma$, then the number of J's such that $\sigma(J) = I$ is $\frac{p!}{p_1! \cdots p_n!}$. Thus by (2.12), we have

$$\begin{aligned}
&\sum_{I,J} \int_{\mathbb{C}^n} A_{I,\overline{J}} z_{i_1} \cdots z_{i_p} \overline{z_{j_1} \cdots z_{j_p}} |z|^{2q} e^{-m|z|^2} dV_0 \\
&= \sum_I A_{I,\overline{I}} \frac{p!}{p_1! \cdots p_n!} \int_{\mathbb{C}^n} |z_1^{p_1} \cdots z_n^{p_n}|^2 |z|^{2q} e^{-m|z|^2} dV_0 \\
&= \sum_I A_{I,\overline{I}} \frac{p!}{p_1! \cdots p_n!} \frac{(n+p+q-1)! p_1! \cdots p_n!}{(p+n-1)! m^{n+q+p}} \qquad \text{by (2.12)} \\
&= (\sum_I A_{I,\overline{I}}) \frac{p!(n+p+q-1)!}{(p+n-1)! m^{n+p+q}}.
\end{aligned}$$

The lemma is proved by observing that

$$e^{-m(\frac{\log m}{\sqrt{m}})^2} = e^{-(\log m)^2} = O(\frac{1}{m^{p'}})$$

for any $p' > 0$.

$\square$

Our prototype of function A is the curvature tensor $(R_{i\bar{j}k\bar{l}})$, which is symmetric. However, in most cases, we encounter functions which are not symmetric. Thus the following corollary is useful.

Corollary 4.1. *Let A be a function on $\{1, \cdots, n\}^p \times \{1, \cdots, n\}^p$ (not necessarily symmetric). Then for $p' > 0$,*

$$\sum_{I,J} \int_{|z| \leq \frac{\log m}{\sqrt{m}}} A_{I,\overline{J}} z_{i_1} \cdots z_{i_p} \overline{z_{j_1} \cdots z_{j_p}} |z|^{2q} e^{-m|z|^2} dV_0$$
$$= (\frac{1}{p!} \sum_I \sum_{\sigma \in \Sigma} A_{I,\overline{\sigma(I)}}) \frac{p!(n+p+q-1)!}{(p+n-1)! m^{n+p+q}} + O(\frac{1}{m^{p'}}).$$

Proof: The symmetrization of A is

$$\tilde{A}_{I,\overline{J}} = \frac{1}{(p!)^2} \sum_{\sigma,\eta \in \Sigma} A_{\sigma(I),\overline{\eta(J)}}.$$

Using Lemma 4.1, we have

$$\sum_{I,J} \int_{|z| \leq \frac{\log m}{\sqrt{m}}} A_{I,\overline{J}} z_{i_1} \cdots z_{i_p} \overline{z_{j_1} \cdots z_{j_p}} |z|^{2q} e^{-m|z|^2} dV_0$$
$$= \sum_{I,J} \int_{|z| \leq \frac{\log m}{\sqrt{m}}} \tilde{A}_{I,\overline{J}} z_{i_1} \cdots z_{i_p} \overline{z_{j_1} \cdots z_{j_p}} |z|^{2q} e^{-m|z|^2} dV_0$$
$$= \sum_I \tilde{A}_{I,\overline{I}} \frac{p!(n+p+q-1)!}{(p+n-1)! m^{n+p+q}} + O(\frac{1}{m^{p'}}).$$

The corollary then follows from the elementary fact that

$$\sum_I \tilde{A}_{I,\overline{I}} = \sum_I \frac{1}{(p!)^2} \sum_{\sigma,\eta \in \Sigma} A_{\sigma(I),\overline{\eta(J)}} = \frac{1}{p!} \sum_I \sum_{\sigma \in \Sigma} A_{\sigma(I),\overline{\sigma(I)}}.$$

$\square$

Proposition 4.1. *We have*

$$\begin{aligned}
K(1) &= \frac{1}{m^n} + O(\frac{1}{m^{n+3}}) \\
mK(e_4) &= \frac{1}{2}\rho\frac{1}{m^{n+1}} + O(\frac{1}{m^{n+3}}) \\
K(c_2) &= -\rho\frac{1}{m^{n+1}} + O(\frac{1}{m^{n+3}}) \\
mK(e_6) &= -\frac{1}{6}(-\Delta\rho + 2|Ric|^2 + |R|^2)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}}) \\
\frac{1}{2}m^2K(e_4^2) &= \frac{1}{8}(\rho^2 + 4|Ric|^2 + |R|^2)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}}) \\
mK(c_2e_4) &= -\frac{1}{2}(\rho^2 + 2|Ric|^2)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}}) \\
\frac{1}{2}K(c_2^2) &= \frac{1}{2}(\rho^2 + |Ric|^2)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}}) \\
K(c_4) &= -\frac{1}{2}(\Delta\rho - |Ric|^2)\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}}).
\end{aligned}$$

Proof: We consider $mK(e_4)$ first. Since

$$e_4 = -\frac{1}{4}R_{i\bar{j}k\bar{l}}z_iz_k\overline{z}_j\overline{z}_l,$$

by using Lemma 4.1, we have

$$mK(e_4) = m(-\frac{1}{4}R_{i\bar{j}i\bar{j}} \cdot \frac{2!}{m^{n+2}}) + O(\frac{1}{m^{n+3}}) = \frac{1}{2}\rho\frac{1}{m^{n+1}} + O(\frac{1}{m^{n+3}}).$$

Similarly, we can prove the formulas for $K(1), K(c_2), mK(e_6)$ and $K(c_4)$. Next we consider $\frac{1}{2}m^2K(e_4^2)$. We have

$$e_4^2 = \frac{1}{16}R_{i\bar{j}k\bar{l}}R_{p\bar{q}r\bar{s}}z_iz_kz_pz_r\overline{z}_j\overline{z}_l\overline{z}_q\overline{z}_s.$$

Using Corollary 4.1, we have

$$\begin{aligned}
\frac{1}{2}m^2K(e_4^2) = \frac{1}{2}\cdot\frac{1}{16}m^2\cdot\frac{1}{6}(&R_{i\bar{i}j\bar{j}}R_{k\bar{k}l\bar{l}} \\
&+ 4R_{i\bar{i}j\bar{k}}R_{k\bar{j}l\bar{l}} + R_{i\bar{j}k\bar{l}}R_{j\bar{i}l\bar{k}})\frac{4!}{m^{n+4}} + O(\frac{1}{m^{n+3}}) \\
= \frac{1}{8}(\rho^2 + 4|Ric|^2 + |R|^2)&\frac{1}{m^{n+2}} + O(\frac{1}{m^{n+3}}).
\end{aligned}$$

The remaining terms can be treated in the similar way.

□

Proof of Theorem 1.1. By Equation (2.1) and (2.2) we know that the left hand side of Equation (1.1) is equal to

$$I_{00}|\lambda_{(0,\cdots,0)}|^2.$$

By Equation (4.1), we see that

$$|\lambda_{(0,\cdots,0)}|^{-2} = \int_{|z|\leq \frac{\log m}{\sqrt{m}}} e^{m\xi+\eta} e^{-m|z|^2} dV_0,$$

where $m\xi + \eta = a^m \det g_{\alpha\overline{\beta}} e^{m|z|^2}$ is a regular series by Lemma 2.3. Thus by Lemma 2.4, we have an asymptotic expansion of $|\lambda_{(0,\cdots,0)}|^{-2}$ of index $(-n)$ whose each term can be represented by the polynomial of the curvature and its derivatives. Combining this fact to Theorem 3.1, we know all the a_i's must be polynomials of the curvature and its derivatives of weight i. These terms can be obtained by finite many steps of algebraic operations. Moreover, by (4.11) and and Proposition 4.1, we see that

$$\begin{aligned} |\lambda_{(0,\cdots,0)}|^{-2} &= \frac{1}{m^n}(1 - \frac{1}{2}\rho\frac{1}{m} \\ &- \frac{1}{m^2}(\frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|Ric|^2 - 3\rho^2)) + O(\frac{1}{m^{5/2}})), \end{aligned} \tag{4.12}$$

from which a_0, a_1, a_2 can be calculated by the above equation as follows:

$$\begin{cases} a_0 = 1 \\ a_1 = \frac{1}{2}\rho \\ a_2 = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|Ric|^2 + 3\rho^2). \end{cases}$$

This completes the proof of Theorem 1.1 except for the a_3 term.

□

5. Computation of the a_3 Term

In this section, we compute the a_3 term. The first step is to compute σ_3 in the expansion of Theorem 3.1.

Theorem 5.1. *With all the notations as Theorem 3.1, we have*

$$\sigma_3 = \frac{1}{4}|D'\rho|^2,$$

where ρ is the scalar curvature of the metric g and $|D'\rho|^2 = \sum|\frac{\partial\rho}{\partial z_i}|^2$ under local normal coordinate system.

Proof. Let

$$V = \{S \in H^0(M, L^m) | S(x_0) = 0, DS(x_0) = 0\}.$$

Then $H^0(M, L^m)$ is spanned by $S_0 = S^{p'}_{(0,\cdots,0),m}, S_1 = S^{p'}_{(1,\cdots,0),m}, \cdots, S_n = S^{p'}_{(0,\cdots,1),m}$ and V, where $\{S^{p'}_{P,m}\}$ are defined in Section 2 as peak sections. Let $T_1, \cdots, T_{d-n-1}$ be an orthonormal basis of V with $d = \dim H^0(M, L^m)$. The metric matrix (F_{AB}) can be represented by block matrix

$$\begin{pmatrix} 1 & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & E \end{pmatrix},$$

where $M_{12} = ((S_0, S_1), \cdots, (S_0, S_n))$, $M_{22} = ((S_i, S_j)(1 \le i, j \le n))$, $M_{13} = ((S_0, T_\alpha), 1 \le \alpha \le d-n-1)$, $M_{31} = \overline{M_{13}^T})$, $M_{23} = ((S_i, T_\alpha), 1 \le i \le n, 1 \le \alpha \le d-n-1)$, E is the unit matrix and $F_{AB} = \overline{F_{BA}}$.

A straightforward computation using Lemma 3.1 shows that

$$I_{00} = 1 + \begin{pmatrix} M_{12} \, M_{13} \end{pmatrix} \tilde{M}^{-1} \begin{pmatrix} M_{21} \\ M_{31} \end{pmatrix},$$

where

$$\tilde{M} = \begin{pmatrix} M_{22} & M_{23} \\ M_{32} & E \end{pmatrix} - \begin{pmatrix} M_{21} \\ M_{31} \end{pmatrix} \begin{pmatrix} M_{12} & M_{13} \end{pmatrix}.$$

By Lemma 2.2, $M_{12} = O(\frac{1}{m^{3/2}})$ and $M_{13} = O(\frac{1}{m^2})$. Thus we have

$$I_{00} = 1 + M_{12}M_{21} + O(\frac{1}{m^{7/2}}) = 1 + \sum_{i=1}^{n} |(S_0, S_i)|^2 + O(\frac{1}{m^{7/2}}). \tag{5.1}$$

By the definition of $S_i (0 \le i \le n)$, we have

$$(S_0, S_i) = \lambda_{(0,\cdots,0)} \lambda_{(0,\cdots,1,\cdots,0)} \int_{|z| \le \frac{\log m}{\sqrt{m}}} \overline{z}_i a^m dV_g + O(\frac{1}{m^2}). \tag{5.2}$$

It is easy to see (cf. [11]) that

$$\begin{aligned} |\lambda_{(0,\cdots,0)}|^{-2} &= \frac{1}{m^n}(1 + O(\frac{1}{m})) \\ |\lambda_{(0,\cdots,1\cdots,0)}|^{-2} &= \frac{1}{m^{n+1}}(1 + O(\frac{1}{m})). \end{aligned} \tag{5.3}$$

On the other hand, since

$$\log a = -|z|^2 + \frac{1}{4} R_{i\bar{j}k\bar{l}} z_i \overline{z}_j z_k \overline{z}_l + O(|z|^5),$$

and since

$$\log \det g_{i\bar{j}} = -R_{i\bar{j}} z_i \overline{z}_j + O(|z|^3),$$

we have

$$\begin{aligned} \int_{|z| \le \frac{\log m}{\sqrt{m}}} \overline{z}_i a^m dV_g &= -\frac{1}{12} m \int_{|z| \le \frac{\log m}{\sqrt{m}}} R_{p\bar{q}r\bar{s},t} \delta_{iu} z_p z_r z_t \overline{z}_q \overline{z}_s \overline{z}_u e^{-m|z|^2} dV_0 \\ &\quad - \frac{1}{2} \int_{|z| \le \frac{\log m}{\sqrt{m}}} R_{p\bar{q},r} \delta_{iu} z_p z_r \overline{z}_q \overline{z}_u e^{-m|z|^2} dV_0 + O(\frac{1}{m^2}). \end{aligned}$$

where dV_0 is the Euclidean volume form.

Thus by Lemma 4.1, we have

$$\int_{|z| \le \frac{\log m}{\sqrt{m}}} \overline{z}_i a^m dV_g = -\frac{1}{2} \frac{\rho}{m^{n+2}} (1 + O(\frac{1}{m})). \tag{5.4}$$

Using (5.1), (5.2), (5.3) and (5.4), we see that

$$\sigma_3 = \frac{1}{4} |D' \rho|^2.$$

$\square$

We now estimate

$$|\lambda_{(0,\cdots,0)}|^{-2} = \int_{|z|\leq\frac{\log m}{\sqrt{m}}} a^m dV_g$$

to the term $\frac{1}{m^{n+3}}$, from which a_3 can be calculated.

We define our notations in the following equations in addition to Equation (4.2)- (4.6).

$$\begin{array}{l} |R|^2 = \sum_{i,j,k,l=1}^n |R_{i\bar{j}k\bar{l}}|^2 \\ |Ric|^2 = \sum_{i,j=1}^n |R_{i\bar{j}}|^2 \\ |D'\rho|^2 = \sum_{i=1}^n |\frac{\partial\rho}{\partial z_i}|^2 \\ |D'Ric|^2 = \sum_{i,j,k=1}^n |R_{i\bar{j},k}|^2 \\ |D'R|^2 = \sum_{i,j,k,l,p=1}^n |R_{i\bar{j}k\bar{l},p}|^2 \\ div\, div\,(\rho Ric) = \sum_{i,j=1}^n (\rho R_{j\bar{i}})_{\bar{j}i} \\ div\, div\,(R, Ric) = \sum_{i,j,k,l=1}^n (R_{i\bar{j}k\bar{l}} R_{j\bar{i}})_{l\bar{k}} \\ R(Ric, Ric) = \sum_{i,j,k,l=1}^n R_{i\bar{j}k\bar{l}} R_{j\bar{i}} R_{l\bar{k}} \\ Ric(R,R) = \sum_{i,j,k,l,p,q=1}^n R_{i\bar{j}} R_{j\bar{k}p\bar{q}} R_{k\bar{i}q\bar{p}} \\ \sigma_1(R) = \sum_{i,j,k,l,p,q=1}^n R_{i\bar{j}k\bar{l}} R_{l\bar{k}p\bar{q}} R_{q\bar{p}j\bar{i}} \\ \sigma_2(R) = \sum_{i,j,k,l,p,q=1}^n R_{i\bar{j}k\bar{l}} R_{p\bar{i}q\bar{k}} R_{j\bar{p}l\bar{q}} \\ \sigma_3(Ric) = \sum_{i,j,k=1}^n R_{i\bar{j}} R_{j\bar{k}} R_{k\bar{i}}, \end{array} \tag{5.5}$$

where "$,p$" represents the covariant derivative in the direction $\frac{\partial}{\partial z_p}$.

In order to estimate $|\lambda_{(0,\cdots,0)}|^{-2}$ up to the term $\frac{1}{m^{n+3}}$, we must use the Taylor expansion of $\log a$ up to degree 8 and the Taylor expansion of $\log\det g_{i\bar{j}}$ up to degree 6. Suppose we have the following expansions at x_0:

$$\begin{aligned} \log a &= -|z|^2 + e_4 + e_5 + e_6 + e_7 + e_8 + O(|z|^9) \\ \log\det(g_{i\bar{j}}) &= c_2 + c_3 + c_4 + c_5 + c_6 + O(|z|^7), \end{aligned} \tag{5.6}$$

where e_4, e_5, e_6, e_7 and e_8 are homogeneous polynomials of z and $\overline{z}$ of degree 4,5,6,7 and 8, respectively and c_2, c_3, c_4 c_5 and c_6 are homogeneous polynomials of degree 2,3,4,5 and 6, respectively.

Considering the function $e^{m(\log a+|z|^2)}e^{\log\det g_{\alpha\overline{\beta}}}$, by (5.6), we have

$$
\begin{aligned}
&e^{m(\log a+|z|^2)}e^{\log\det g_{i\bar{j}}}\\
&= e^{m(e_4+e_5+e_6+e_7+e_8+O(|z|^9))}e^{c_2+c_3+c_4+c_5+c_6+O(|z|^7)}\\
&= 1+m(e_4+e_5+e_6+e_7+e_8)+\frac{1}{2}m^2(e_4^2+e_5^2+2e_4e_5+2e_4e_6)\\
&+\frac{1}{6}m^3e_4^3+c_2+mc_2(e_4+e_5+e_6)+\frac{1}{2}m^2c_2e_4^2+c_3+mc_3e_5+c_4\\
&+mc_4e_4+c_5+c_6+\frac{1}{2}mc_2^2e_4+\frac{1}{2}(c_2^2+c_3^2+2c_2c_3+2c_2c_4)\\
&+\frac{1}{6}c_2^3+O(\cdots),
\end{aligned}
$$

where $O(\cdots)$ represents the sum of terms, each of which is less than a constant multiple of $m^\mu|z|^t$ for some μ and t such that $t-2\mu>6$. Those terms will not affect the value of a_3, and can be omitted.

Let $K(\varphi)$ be the functional defined in Equation (4.10). We have $K(e_5)=K(e_7)=K(c_2)=K(c_5)=K(c_2e_5)=K(c_2c_3)=0$. Thus

$$
\begin{aligned}
&|\lambda_{(0,\cdots,0)}|^{-2}=K(e^{m(|z|^2+\log a)}e^{\log\det g_{i\bar{j}}})\\
&=K(1)+mK(e_4)+mK(e_6)+mK(e_8)+\frac{1}{2}m^2K(e_4^2)\\
&+\frac{1}{2}m^2K(e_5^2)+m^2K(e_4e_6)+\frac{1}{6}m^3K(e_4^3)+K(c_2)+mK(c_2e_4)\\
&+mK(c_2e_6)+\frac{1}{2}m^2K(c_2e_4^2)+mK(c_3e_5)+K(c_4)+mK(c_4e_4)\\
&+K(c_6)+\frac{1}{2}mK(c_2^2e_4)+\frac{1}{2}K(c_2^2)+\frac{1}{2}K(c_3^2)+K(c_2c_4)\\
&+\frac{1}{6}K(c_2^3)+O(\cdots).
\end{aligned}
\tag{5.7}
$$

Suppose A is a homogeneous polynomial on $\mathbb{C}^n$:

$$
A=\sum_{I,J}A_{I,\overline{J}}z_I\overline{z}_J,
$$

where $z_I=z_{i_1}\cdots z_{i_p}$ and $z_J=z_{j_1}\cdots z_{j_p}$ for $I=(i_1,\cdots,i_p)$ and $J=(j_1,\cdots,j_p)$. Define

$$
L(A)=\frac{1}{p!}\sum_I\sum_{\sigma\in\Sigma}A_{I,\overline{\sigma(I)}}.
$$

Sometimes we also use $L(A_{I,\overline{I}})$ to denote $L(A)$. For example, $L(R_{i\bar{j}k\bar{l}}z_iz_k\overline{z}_j\overline{z}_l)=L(\sum R_{i\bar{i}j\bar{j}})=-\rho$, the scalar curvature. By Lemma 4.1

and Equation (5.7), we have

(5.8)

$$|\lambda_{(0,\cdots,0)}|^{-2} = \frac{1}{m^n}(1 + \frac{1}{m}(2L(e_4) + L(c_2))$$
$$+ \frac{1}{m^2}(6L(\tilde{e}_6) + 12L(e_4^2) + 6L(c_2e_4) + L(c_2^2) + 2L(\tilde{c}_4))$$
$$+ \frac{1}{m^3}(L(c_2^3) + 3L(c_3^2) + 6L(c_2\tilde{c}_4) + 24L(\tilde{c}_4e_4) + 12L(c_2^2e_4) + 60L(c_2e_4^2)$$
$$+ 120L(e_4^3) + 120L(e_4\tilde{e}_6) + 24L(c_2\tilde{e}_6) + 24L(c_3e_5) + 60L(e_5^2)$$
$$+ 6L(\tilde{c}_6) + 24L(\tilde{e}_8)) + O(\frac{1}{m^4})).$$

Proposition 5.1. *With all the notations as above, we have*

$$L(c_2^3) + 3L(c_3^2) + 6L(c_2\tilde{c}_4) + 24L(\tilde{c}_4e_4) + 12L(c_2^2e_4) + 60L(c_2e_4^2)$$
$$+ 120L(e_4^3) + 120L(e_4\tilde{e}_6) + 24L(c_2\tilde{e}_6) + 24L(c_3e_5) + 60L(e_5^2)$$
$$+ 6L(\tilde{c}_6) + 24L(\tilde{e}_8)$$
$$= -\frac{1}{8}\Delta\Delta\rho + \frac{1}{4}|D'\rho|^2 + \frac{1}{4}|D'Ric|^2 - \frac{1}{24}|D'R|^2 + \frac{1}{6}\rho\Delta\rho + \frac{3}{8}R_{i\bar{j}}\rho_{j\bar{i}} \tag{5.9}$$
$$- \frac{1}{48}\rho^3 - \frac{1}{12}\rho|Ric|^2 + \frac{1}{48}\rho|R|^2 - \frac{1}{12}\sigma_1(R) + \frac{1}{24}\sigma_2(R) + \frac{1}{6}\sigma_3(Ric)$$
$$+ \frac{1}{4}R(Ric, Ric).$$

We postpone the proof of this proposition to the Appendix of this paper.

□

Proposition 5.2. *We have*

$$div\, div\,(R, Ric) = -R_{i\bar{j}}\rho_{j\bar{i}} - 2|D'Ric|^2 + R_{j\bar{i}l\bar{k}}R_{i\bar{j},k\bar{l}}$$
$$- R(Ric, Ric) - \sigma_3(Ric)$$
$$div\, div\,(\rho Ric) = 2|D'\rho|^2 + R_{i\bar{j}}\rho_{j\bar{i}} + \rho\Delta\rho$$
$$\Delta|R|^2 = -2R_{j\bar{i}l\bar{k}}R_{i\bar{j},k\bar{l}} + 2|D'R|^2 + 4\sigma_1(R) - 2\sigma_2(R) + 2Ric(R, R)$$
$$\Delta|Ric|^2 = 2|D'Ric|^2 + 2R_{i\bar{j}}\rho_{j\bar{i}} + 2R(Ric, Ric) + 2\sigma_3(Ric)$$
$$\Delta\rho^2 = 2|D'\rho|^2 + 2\rho\Delta\rho.$$

Proof: A straightforward computation.

□

From (4.12), we know

$$|\lambda_{(0,\cdots,0)}|^{-2} = \frac{1}{m^n}(1 - \frac{1}{2}\rho\frac{1}{m}$$
$$- (\frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|Ric|^2 - 3\rho^2))\frac{1}{m^2} + O(\frac{1}{m^{5/2}}).$$

The term a_3 can be computed from Proposition 5.1, Proposition 5.2, Theorem 5.1, Equation (2.1), (2.2), (5.8) and the above expression.

□

Example 1. *Let $M = CP^n$, $L = \mathcal{O}(1)$ be the hyperplane bundle. For any m,*

$$\sqrt{\frac{(m+n)!}{P!}}z^P$$

for $P \in \mathbb{Z}_+^n$ with $|P| = m$ form an orthonormal basis of $H^0(M, L^m)$. Using this we see that

$$\sum_{|P|=m} ||\sqrt{\frac{(m+n)!}{P!}}z^P||_{h_m}^2 = \frac{(m+n)!}{m!}$$
$$= m^n(1 + \frac{1}{2}n(n+1)\frac{1}{m} + \frac{1}{24}n(n+1)(n-1)(3n+2)\frac{1}{m^2}$$
$$+ \frac{1}{48}n^2(n+1)^2(n-1)(n-2)\frac{1}{m^3} + O(\frac{1}{m^4})).$$

Corollary 5.1. *Riemann-Roch Theorem can be recovered from Theorem 1.1, at least asymptotically. Integration against M on both side of* (1.1) *gives*

$$\dim H^0(M, L^m) = m^n(vol(M) + \frac{1}{2}c_1(M)\frac{1}{m}$$
$$+ \frac{1}{12}(c_2 + c_1^2)\frac{1}{m^2} + \frac{1}{24}c_1c_2\frac{1}{m^3} + O(\frac{1}{m^4})).$$

□

6. Appendix

In this Appendix we prove Proposition 5.1. Using the notations as in the previous sections, it is splitted into the following 13 claims.

Claim 1.

$$L(c_2^3) = -\frac{1}{6}(\rho^3 + 3\rho|Ric|^2 + 2\sigma_3(Ric)).$$

Proof:

$$L(c_2^3) = -\frac{1}{6}(R_{i\bar{i}}R_{k\bar{k}}R_{p\bar{p}} + R_{i\bar{i}}R_{k\bar{p}}R_{p\bar{k}} + R_{i\bar{k}}R_{k\bar{i}}R_{p\bar{p}}$$
$$+ R_{i\bar{k}}R_{k\bar{p}}R_{p\bar{i}} + R_{i\bar{p}}R_{k\bar{i}}R_{p\bar{k}} + R_{i\bar{p}}R_{k\bar{k}}R_{p\bar{i}})$$
$$= -\frac{1}{6}(\rho^3 + 3\rho|Ric|^2 + 2\sigma_3(Ric)).$$

□

Claim 2.

$$3L(c_3^2) = |D'\rho|^2 + \frac{1}{2}|D'Ric|^2.$$

Proof:

$$\begin{aligned}3L(c_3^2) &= \frac{3}{2}L(R_{i\bar{i},k}R_{p\overline{p},\overline{k}})\\ &= \frac{1}{2}(R_{i\bar{i},k}R_{p\overline{p},\overline{k}} + R_{i\overline{p},k}R_{p\bar{i},\overline{k}} + R_{i\overline{k},k}R_{p\overline{p},\bar{i}})\\ &= \frac{1}{2}(2|D'\rho|^2 + |D'Ric|^2).\end{aligned}$$

□

Claim 3.

$$12L(c_2^2e_4) = \frac{1}{4}(\rho^3 - 2R(Ric, Ric) + 4\sigma_3(Ric) + 5\rho|Ric|^2).$$

Proof:

$$\begin{aligned}&12L(c_2^2e_4) = -3L(R_{i\bar{i}k\overline{k}}R_{p\overline{p}}R_{r\overline{r}})\\ &= -\frac{1}{4}(R_{i\bar{i}k\overline{k}}R_{p\overline{p}}R_{r\overline{r}} + R_{i\bar{i}k\overline{k}}R_{p\overline{r}}R_{r\overline{p}} + R_{i\bar{i}k\overline{p}}R_{p\overline{k}}R_{r\overline{r}} + R_{i\bar{i}k\overline{p}}R_{p\overline{r}}R_{r\overline{k}}\\ &+ R_{i\bar{i}k\overline{r}}R_{p\overline{k}}R_{r\overline{p}} + R_{i\bar{i}k\overline{r}}R_{p\overline{p}}R_{r\overline{k}} + R_{i\overline{k}k\overline{p}}R_{p\bar{i}}R_{r\overline{r}} + R_{i\overline{k}k\overline{p}}R_{p\overline{r}}R_{r\bar{i}}\\ &+ R_{i\overline{k}k\overline{r}}R_{p\bar{i}}R_{r\overline{p}} + R_{i\overline{k}k\overline{r}}R_{p\overline{p}}R_{r\bar{i}} + R_{i\overline{p}k\overline{r}}R_{p\bar{i}}R_{r\overline{k}} + R_{i\overline{p}k\overline{r}}R_{p\overline{k}}R_{r\bar{i}})\\ &= \frac{1}{4}(\rho^3 - 2R(Ric, Ric) + 4\sigma_3(Ric) + 5\rho|Ric|^2).\end{aligned}$$

□

Claim 4.

$$6L(c_2\tilde{c}_4) = \frac{1}{2}\rho\Delta\rho + R_{i\bar{j}}\rho_{j\bar{i}} - \frac{1}{2}\rho|Ric|^2 + R(Ric, Ric).$$

Proof:

$$\begin{aligned}6L(c_2\tilde{c}_4) &= \frac{3}{2}L(R_{p\overline{p}}(R_{i\bar{i},k\overline{k}} + R_{i\overline{r}k\overline{k}}R_{r\bar{i}}))\\ &= \frac{1}{2}(R_{p\overline{p}}(R_{i\bar{i},k\overline{k}} + R_{i\overline{r}k\overline{k}}R_{r\bar{i}}) + 2R_{p\overline{k}}(R_{i\bar{i},k\overline{p}} + R_{i\overline{r}k\overline{p}}R_{r\bar{i}}))\\ &= \frac{1}{2}(\rho\Delta\rho - \rho|Ric|^2 + 2R_{p\overline{k}}\rho_{k\overline{p}} + 2R(Ric, Ric)).\end{aligned}$$

□

Claim 5.

$$\begin{aligned}120L(e_4^3) = \frac{1}{48}\rho^3 + \frac{1}{4}\rho|Ric|^2 + \frac{1}{16}\rho|R|^2 - \frac{1}{2}R(Ric, Ric)\\ + \frac{1}{2}Ric(R, R) - \frac{1}{6}\sigma_1(R) - \frac{1}{24}\sigma_2(R) + \frac{1}{3}\sigma_3(Ric).\end{aligned}$$

Proof: Suppose

$$A_{k\overline{k}_1l\overline{l}_1p\overline{p}_1r\overline{r}_1} = L(R_{k\overline{k_1}l\overline{l_1}}R_{p\overline{p_1}r\overline{r_1}}).$$

Noting that $A_{k\overline{k}_1l\overline{l}_1p\overline{p}_1r\overline{r}_1}$ is *not* symmetric, we have

$$L(R_{i\bar i j\bar j}A_{k\bar k l\bar l p\bar p r\bar r}) = \frac{1}{15}(R_{i\bar i j\bar j}A_{k\bar k l\bar l p\bar p r\bar r} + 8R_{i\bar i j\bar k}A_{k\bar j l\bar l p\bar p r\bar r}$$
$$+ 2R_{i\bar j k\bar l}A_{j\bar i l\bar k p\bar p r\bar r} + 4R_{i\bar j k\bar l}A_{j\bar i p\bar p l\bar k r\bar r}). \tag{6.1}$$

We compute the above expression term by term as follows:

$$A_{k\bar k l\bar l p\bar p r\bar r} = \frac{1}{6}(R_{k\bar k l\bar l}R_{p\bar p r\bar r} + 4R_{k\bar r l\bar l}R_{p\bar p r\bar k} + R_{k\bar r l\bar p}R_{p\bar k r\bar l})$$
$$= \frac{1}{6}(\rho^2 + 4|Ric|^2 + |R|^2). \tag{6.2}$$

$$R_{i\bar i j\bar k}A_{k\bar j l\bar l p\bar p r\bar r} = -R_{j\bar k}L(R_{k\bar j l\bar l}R_{p\bar p r\bar r})$$
$$= -\frac{1}{6}R_{j\bar k}(R_{k\bar j l\bar l}R_{p\bar p r\bar r} + R_{k\bar j l\bar p}R_{p\bar l r\bar r} + R_{k\bar j l\bar r}R_{p\bar p r\bar l}$$
$$+ R_{k\bar l l\bar p}R_{p\bar j r\bar r} + R_{k\bar l l\bar r}R_{p\bar j r\bar p} + R_{k\bar p l\bar r}R_{p\bar j r\bar l})$$
$$= -\frac{1}{6}\rho|Ric|^2 + \frac{1}{3}R(Ric, Ric) - \frac{1}{3}\sigma_3(Ric) - \frac{1}{6}Ric(R, R). \tag{6.3}$$

$$R_{i\bar j k\bar l}A_{j\bar i l\bar k p\bar p r\bar r} = R_{i\bar j k\bar l}L(R_{j\bar i l\bar k}R_{p\bar p r\bar r})$$
$$= \frac{1}{6}R_{i\bar j k\bar l}(R_{j\bar i l\bar k}R_{p\bar p r\bar r} + R_{j\bar i l\bar p}R_{p\bar k r\bar r} + R_{j\bar i l\bar r}R_{p\bar k r\bar p}$$
$$+ R_{j\bar k l\bar p}R_{p\bar i r\bar r} + R_{j\bar k l\bar r}R_{p\bar i r\bar p} + R_{j\bar p l\bar r}R_{p\bar i r\bar k})$$
$$= -\frac{1}{6}\rho|R|^2 - \frac{2}{3}Ric(R, R) + \frac{1}{6}\sigma_2(R). \tag{6.4}$$

$$R_{i\bar j k\bar l}A_{j\bar i p\bar p l\bar k r\bar r} = R_{i\bar j k\bar l}L(R_{j\bar i p\bar p}R_{l\bar k r\bar r})$$
$$= \frac{1}{6}R_{i\bar j k\bar l}(R_{j\bar i p\bar p}R_{l\bar k r\bar r} + R_{j\bar i p\bar k}R_{l\bar p r\bar r} + R_{j\bar i p\bar r}R_{l\bar p r\bar k}$$
$$+ R_{j\bar p p\bar k}R_{l\bar i r\bar r} + R_{j\bar p p\bar r}R_{l\bar i r\bar k} + R_{j\bar k p\bar r}R_{l\bar i r\bar p})$$
$$= \frac{1}{3}R(Ric, Ric) - \frac{1}{3}Ric(R, R) + \frac{1}{3}\sigma_1(R). \tag{6.5}$$

Thus by (6.2) through (6.5), we have

$$L(R_{i\bar i j\bar j}A_{k\bar k l\bar l p\bar p r\bar r}) = \frac{1}{15}(-\frac{1}{6}\rho^3 - 2\rho|Ric|^2 - \frac{1}{2}\rho|R|^2 + 4R(Ric, Ric)$$
$$- 4Ric(R, R) + \frac{4}{3}\sigma_1(R) + \frac{1}{3}\sigma_2(R) - \frac{8}{3}\sigma_3(Ric)).$$

□

Claim 6.

$$24L(\tilde c_4 e_4) = -\frac{1}{4}\rho\Delta\rho + \frac{1}{4}\rho|Ric|^2 - R_{i\bar p}\rho_{p\bar i}$$
$$- R(Ric, Ric) + \frac{1}{4}R_{i\bar p k\bar r}R_{p\bar i, r\bar k} + \frac{1}{4}Ric(R, R).$$

Proof:

$$
\begin{aligned}
&24L(\tilde{c}_4 e_4) = 24L((\tilde{c}_4)_{i\bar{i}k\bar{k}}(e_4)_{p\bar{p}r\bar{r}}) \\
&= 4(c_4)_{i\bar{i}k\bar{k}}(e_4)_{p\bar{p}r\bar{r}} + 16(\tilde{c}_4)_{i\bar{p}k\bar{k}}(e_4)_{p\bar{i}r\bar{r}} + 4(\tilde{c}_4)_{i\bar{p}k\bar{r}}(e_4)_{p\bar{i}r\bar{k}} \\
&= \frac{1}{4}(R_{i\bar{i}k\bar{k}}(R_{p\bar{p},r\bar{r}} + R_{p\bar{\alpha}r\bar{r}}R_{\alpha\bar{p}}) + R_{i\bar{p}k\bar{k}}(R_{r\bar{r},p\bar{i}} + R_{r\bar{\alpha}p\bar{i}}R_{\alpha\bar{r}}) \\
&+ \frac{1}{4}R_{i\bar{p}k\bar{r}}(R_{p\bar{i},r\bar{k}} + R_{p\bar{\alpha}r\bar{k}}R_{\alpha\bar{i}})) \\
&= -\frac{1}{4}\rho\Delta\rho + \frac{1}{4}\rho|Ric|^2 - R_{i\bar{p}}\rho_{p\bar{i}} \\
&- R(Ric, Ric) + \frac{1}{4}R_{i\bar{p}k\bar{r}}R_{p\bar{i},r\bar{k}} + \frac{1}{4}Ric(R, R).
\end{aligned}
$$

□

Claim 7.

$$
\begin{aligned}
24L(c_2\tilde{e}_6) = &-\frac{1}{6}\rho\Delta\rho - \frac{1}{2}R_{r\bar{i}}\rho_{i\bar{r}} + \frac{1}{3}\rho|Ric|^2 \\
&+ \frac{1}{6}\rho|R|^2 - R(Ric, Ric) + \frac{1}{2}Ric(R, R).
\end{aligned}
$$

Proof:

$$
\begin{aligned}
&24L(c_2\tilde{e}_6) = -24(R_{r\bar{r}}(\tilde{e}_6)_{i\bar{i}k\bar{k}p\bar{p}}) \\
&= -6(R_{r\bar{r}}(\tilde{e}_6)_{i\bar{i}k\bar{k}p\bar{p}} + 3R_{r\bar{i}}(\tilde{e}_6)_{k\bar{k}p\bar{p}i\bar{r}}) \\
&= \frac{1}{6}\rho(-\Delta\rho + 2|Ric|^2 + |R|^2) \\
&+ \frac{1}{2}(R_{r\bar{i}}(-\rho_{i\bar{r}} + R_{k\bar{s}i\bar{r}}R_{s\bar{k}p\bar{p}} + R_{p\bar{s}i\bar{r}}R_{s\bar{p}k\bar{k}} + R_{k\bar{s}p\bar{r}}R_{s\bar{k}i\bar{p}})).
\end{aligned}
$$

□

Claim 8.

$$
\begin{aligned}
120L(e_4\tilde{e}_6) = &\frac{1}{12}\rho\Delta\rho + \frac{1}{2}R_{k\bar{r}}\rho_{r\bar{k}} - \frac{1}{4}R_{i\bar{p}k\bar{r}}R_{p\bar{i},r\bar{k}} \\
&- \frac{1}{6}\rho|Ric|^2 - \frac{1}{12}\rho|R|^2 + R(Ric, Ric) - \frac{3}{4}Ric(R, R) + \frac{1}{2}\sigma_1(R).
\end{aligned}
$$

Proof:

$$
\begin{aligned}
&L(e_4\tilde{e}_6) = \frac{1}{10}((e_4)_{i\bar{i}k\bar{k}}(\tilde{e}_6)_{l\bar{l}p\bar{p}r\bar{r}} + 6(e_4)_{r\bar{i}j\bar{j}}(\tilde{e}_6)_{k\bar{k}p\bar{p}i\bar{r}} + 3(e_4)_{i\bar{p}k\bar{r}}(\tilde{e}_6)_{l\bar{l}p\bar{i}r\bar{k}}) \\
&= \frac{1}{1440}(-\rho(-\Delta\rho + 2|Ric|^2 + |R|^2) \\
&+ 6(-R_{r\bar{i}})(-\rho_{r\bar{k}} + R_{k\bar{s}i\bar{r}}R_{s\bar{k}p\bar{p}} + R_{p\bar{s}i\bar{r}}R_{s\bar{p}k\bar{k}} + R_{k\bar{s}p\bar{r}}R_{s\bar{k}i\bar{p}}) \\
&+ 3R_{i\bar{p}k\bar{r}}(-R_{p\bar{i},r\bar{k}} + R_{l\bar{s}r\bar{k}}R_{s\bar{l}p\bar{i}} + R_{p\bar{s}r\bar{k}}R_{s\bar{i}l\bar{l}} + R_{l\bar{s}p\bar{k}}R_{s\bar{l}r\bar{i}})) \\
&= \frac{1}{1440}(\rho\Delta\rho - 2\rho|Ric|^2 - \rho|R|^2 + 6R_{k\bar{r}}\rho_{r\bar{k}} + 12R(Ric, Ric) \\
&- 6Ric(R, R) - 3R_{i\bar{p}k\bar{r}}R_{p\bar{i},r\bar{k}} + 3\sigma_1(R) - 3Ric(R, R) + 3\sigma_1(R)).
\end{aligned}
$$

$\square$

Claim 9.

$$60L(c_2e_4^2) = -\frac{1}{8}\rho^3 - \rho|Ric|^2 - \frac{1}{8}\rho|R|^2$$
$$+ R(Ric, Ric) - \sigma_3(Ric) - \frac{1}{2}Ric(R, R).$$

Proof: We use the definition of $A_{k\bar{k}l\bar{l}p\bar{p}r\bar{r}}$ in Claim 5.

$$L(c_2e_4^2) = -\frac{1}{80}(R_{i\bar{i}}A_{k\bar{k}l\bar{l}p\bar{p}r\bar{r}} + 4R_{j\bar{k}}A_{k\bar{j}l\bar{l}p\bar{p}r\bar{r}})$$
$$= -\frac{1}{80}(\frac{1}{6}\rho^3 + \frac{4}{3}\rho|Ric|^2 + \frac{1}{6}\rho|R|^2$$
$$-\frac{4}{3}R(Ric, Ric) + \frac{2}{3}Ric(R, R) + \frac{4}{3}\sigma_3(Ric)).$$

$\square$

Claim 10.

$$24L(c_3e_5) = -|D'\rho|^2 - |D'Ric|^2.$$

Proof:

$$24L(c_3e_5) = (R_{i\bar{i},k}R_{p\bar{p}r\bar{r},\bar{k}} + R_{i\bar{p},k}R_{p\bar{i}r\bar{r},\bar{k}})$$
$$= -|D'\rho|^2 - |D'Ric|^2.$$

$\square$

Claim 11.

$$60L(e_5^2) = \frac{1}{12}(3|D'\rho|^2 + 6|D'Ric|^2 + |D'R|^2).$$

Proof:

$$60L(e_5^2) = \frac{120}{144}L(R_{i\bar{i}j\bar{j},k}R_{p\bar{p}r\bar{r},\bar{k}})$$
$$= \frac{1}{12}(3R_{i\bar{i}j\bar{j},k}R_{p\bar{p}r\bar{r},\bar{k}} + 6R_{i\bar{i}j\bar{p},k}R_{p\bar{j}r\bar{r},\bar{k}} + R_{i\bar{p}j\bar{r},k}R_{p\bar{i}r\bar{j},\bar{k}}).$$

$\square$

Claim 12.

$$6L(\tilde{c}_6) = -\frac{1}{6}\Delta\Delta\rho + \frac{2}{3}R_{i\bar{j}}\rho_{j\bar{i}} - \frac{1}{3}R_{j\bar{i}p\bar{k}}R_{i\bar{j},k\bar{p}}$$
$$+ \frac{2}{3}|D'Ric|^2 - \frac{1}{6}Ric(R, R) + \frac{2}{3}R(Ric, Ric) + \frac{1}{3}\sigma_3(Ric).$$

Proof: Suppose we have the Taylor expansion of the function $\log a$ under the local K-coordinates:

$$\log a = -|z|^2 + (e_4)_{i\bar{j}k\bar{l}}z_iz_k\overline{z}_j\overline{z}_l + (e_5^{(1)})_{i\bar{j}k\bar{l}p}z_iz_kz_p\overline{z}_j\overline{z}_l$$
$$+ (e_5^{(2)})_{i\bar{j}k\bar{l}\bar{q}}z_iz_k\overline{z}_j\overline{z}_l\overline{z}_q + (\tilde{e}_6)_{i\bar{j}k\bar{l}p\bar{q}}z_iz_kz_p\overline{z}_j\overline{z}_l\overline{z}_q + O(|z|^7)$$
$$+ (2, 4) + (4, 2),$$

where $(r,s)\in\mathbb{N}$ represents the sum of terms of the form $az_I\overline{z}_J$ such that $|I|=r,|J|=s$. Those terms are irrelevant to the computation of a_3 and need not to be written out explicitly. Thus we have

$$\begin{aligned}g_{i\bar{j}} &= \delta_{ij}-4(e_4)_{i\bar{j}k\bar{l}}z_k\overline{z}_l-6(e_5^{(1)})_{i\bar{j}k\bar{l}p}z_kz_p\overline{z}_l-6(e_5^{(2)})_{i\bar{j}k\bar{l}\bar{q}}z_k\overline{z}_l\overline{z}_q\\ &-9(e_6)_{i\bar{j}k\bar{l}p\bar{q}}z_kz_p\overline{z}_l\overline{z}_q+O(|z|^5)+(1,3)+(3,1).\end{aligned}$$

Thus the inverse matrix $g^{i\bar{j}}$ satisfies

$$\begin{aligned}g^{i\bar{j}} &= \delta_{ij}+4(e_4)_{j\bar{i}k\bar{l}}z_k\overline{z}_l+6(e_5^{(1)})_{j\bar{i}k\bar{l}p}z_kz_p\overline{z}_l+6(e_5^{(2)})_{j\bar{i}k\bar{l}\bar{q}}z_k\overline{z}_l\overline{z}_q\\ &+9(\tilde{e}_6)_{j\bar{i}k\bar{l}p\bar{q}}z_kz_p\overline{z}_l\overline{z}_q+16(e_4)_{j\bar{s}p\bar{q}}(e_4)_{s\bar{i}k\bar{l}}z_p\overline{z}_qz_k\overline{z}_l\\ &+O(|z|^5)+(1,3)+(3,1).\end{aligned}\tag{6.6}$$

On the other hand

$$\begin{aligned}\log\det g_{i\bar{j}} &= (c_2)_{i\bar{j}}z_i\overline{z}_j+(c_3^{(1)})_{i\bar{j}k}z_iz_k\overline{z}_j+(c_3^{(2)})_{i\bar{j}\bar{l}}z_i\overline{z}_j\overline{z}_l\\ &+(\tilde{c}_4)_{i\bar{j}k\bar{l}}z_iz_k\overline{z}_j\overline{z}_l+(c_5^{(1)})_{i\bar{j}k\bar{l}p}z_iz_kz_p\overline{z}_j\overline{z}_l\\ &+(c_5^{(2)})_{i\bar{j}k\bar{l}\bar{q}}z_iz_k\overline{z}_j\overline{z}_l\overline{z}_q+(\tilde{c}_6)_{i\bar{j}k\bar{l}p\bar{q}}z_iz_kz_p\overline{z}_j\overline{z}_l\overline{z}_q+O(|z|^7)\\ &+(1,3)+(3,1)+(1,4)+(4,1)+(1,5)+(5,1)+(2,4)+(4,2).\end{aligned}$$

Consequently

$$\begin{aligned}-R_{i\bar{j}} &= \partial_i\overline{\partial}_j\log\det(g_{\alpha\bar{\beta}})=(c_2)_{i\bar{j}}+2(c_3^{(1)})_{i\bar{j}k}z_k+2(c_3^{(2)})_{i\bar{j}\bar{l}}\overline{z}_l\\ &+4(\tilde{c}_4)_{i\bar{j}k\bar{l}}z_k\overline{z}_l+6(c_5^{(1)})_{i\bar{j}k\bar{l}p}z_kz_p\overline{z}_l+6(c_5^{(2)})_{i\bar{j}k\bar{l}\bar{q}}z_k\overline{z}_l\overline{z}_q\\ &+9(\tilde{c}_6)_{i\bar{j}k\bar{l}p\bar{q}}z_kz_p\overline{z}_l\overline{z}_q+O(|z|^5)+(0,2)+(2,0)+(0,3)+(3,0)\\ &+(0,4)+(4,0)+(1,3)+(3,1).\end{aligned}\tag{6.7}$$

Using (6.6) and (6.7), we have

$$\begin{aligned}&-\frac{1}{4}\frac{\partial^4\rho}{\partial z^k\partial\overline{z}^l\partial z^p\partial\overline{z}^q}z_kz_p\overline{z}_l\overline{z}_q=9(\tilde{c}_6)_{i\bar{i}k\bar{l}p\bar{q}}z_kz_p\overline{z}_l\overline{z}_q+16(e_4)_{j\bar{i}k\bar{l}}(\tilde{c}_4)_{i\bar{j}p\bar{q}}z_k\overline{z}_lz_p\overline{z}_q\\ &+9(c_2)_{i\bar{j}}(\tilde{e}_6)_{j\bar{i}k\bar{l}p\bar{q}}z_k\overline{z}_lz_p\overline{z}_q+12(c_3^{(1)})_{i\bar{j}p}(e_5^{(2)})_{j\bar{i}k\bar{l}\bar{q}}z_k\overline{z}_lz_p\overline{z}_q\\ &+12(c_3^{(2)})_{i\bar{j}\bar{q}}(e_5^{(1)})_{j\bar{i}k\bar{l}p}z_kz_p\overline{z}_l\overline{z}_q+16(c_2)_{i\bar{j}}(e_4)_{j\bar{s}p\bar{q}}(e_4)_{s\bar{i}k\bar{l}}z_p\overline{z}_qz_k\overline{z}_l.\end{aligned}$$

Thus

$$\begin{aligned}&-\frac{1}{4}\frac{\partial^4\rho}{\partial z^k\partial\overline{z}^k\partial z^p\partial\overline{z}^p}=9(\tilde{c}_6)_{i\bar{i}k\bar{k}p\bar{p}}-\frac{3}{4}R_{j\bar{i}}\rho_{i\bar{j}}+\frac{1}{2}R_{j\bar{i}p\bar{k}}R_{i\bar{j},k\bar{p}}-|D'Ric|^2\\ &+\frac{1}{4}Ric(R,R)-R(Ric,Ric)-\frac{1}{2}\sigma_3(Ric).\end{aligned}$$

Since

$$\Delta\Delta\rho=\frac{\partial^2}{\partial z^k\partial\overline{z}^k}(g^{p\bar{q}}\frac{\partial^2\rho}{\partial z^p\partial\overline{z}^q})=\frac{\partial^4\rho}{\partial z^k\partial\overline{z}^k\partial z^p\partial\overline{z}^p}+R_{q\bar{p}}\rho_{p\bar{q}}.$$

We have

$$\frac{1}{4}R_{i\bar{j}}\rho_{j\bar{i}} - \frac{1}{4}\Delta\Delta\rho = 9(\tilde{c}_6)_{i\bar{i}k\bar{k}p\bar{p}} - \frac{3}{4}R_{i\bar{j}}\rho_{j\bar{i}} + \frac{1}{2}R_{\bar{j}i\bar{p}k}R_{i\bar{j},k\bar{p}}$$
$$- |D'Ric|^2 + \frac{1}{4}Ric(R,R) - R(Ric,Ric) - \frac{1}{2}\sigma_3(Ric).$$

Thus

$$(\tilde{c}_6)_{i\bar{i}k\bar{k}p\bar{p}} = -\frac{1}{36}\Delta\Delta\rho + \frac{1}{9}R_{i\bar{j}}\rho_{i\bar{j}} - \frac{1}{18}R_{\bar{j}i\bar{p}k}R_{i\bar{j},k\bar{p}}$$
$$+ \frac{1}{9}|D'Ric|^2 - \frac{1}{36}Ric(R,R) + \frac{1}{9}R(Ric,Ric) + \frac{1}{18}\sigma_3(Ric).$$

□

Claim 13.

$$24L(\tilde{e}_8) = \frac{1}{24}\Delta\Delta\rho - \frac{5}{12}|D'Ric|^2 - \frac{1}{8}|D'R|^2 - \frac{7}{24}R_{i\bar{j}}\rho_{j\bar{i}} + \frac{1}{3}R_{j\bar{i}k\bar{p}}R_{i\bar{j},p\bar{k}}$$
$$+ \frac{1}{6}Ric(R,R) - \frac{5}{12}R(Ric,Ric) - \frac{5}{12}\sigma_1(R) + \frac{1}{12}\sigma_2(R) - \frac{1}{6}\sigma_3(Ric).$$

Proof: A tedious but straightforward computation shows that

$$\frac{\partial^6 \log\det g_{i\bar{j}}}{\partial z^k \partial \overline{z}^k \partial z^p \partial \overline{z}^p \partial z^r \partial \overline{z}^r} = \frac{\partial^6 g_{i\bar{i}}}{\partial z^p \partial \overline{z}^p \partial z^k \partial \overline{z}^k \partial z^r \partial \overline{z}^r}$$
$$- 6R_{j\bar{i}k\bar{p}}\frac{\partial^4 g_{i\bar{j}}}{\partial z^p \partial \overline{z}^l \partial z^r \partial \overline{z}^r} - 6|D'Ric|^2 - 3|D'R|^2 + 6Ric(R,R)$$
$$+ 2\sigma_1(R) + 2\sigma_2(R) - 2\sigma_3(Ric).$$

Thus using Claim 12,

$$24^2 e_8 = \Delta\Delta\rho - 10|D'Ric|^2 - 7R_{i\bar{j}}\rho_{j\bar{i}} + 8R_{j\bar{i}k\bar{p}}R_{i\bar{j},p\bar{k}}$$
$$- 3|D'R|^2 + 4Ric(R,R) - 10R(Ric,Ric)$$
$$- 10\sigma_1(R) + 2\sigma_2(R) - 4\sigma_3(Ric).$$

□

Proposition 5.1 follows from the above claims.

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